ELEMENTS OF

Mechanical Vibration

BY CARL ROGER FREBERG

AND

EMORY N. KEMLER

ELEMENTS OF MECHANICAL VIBRATION

Second Edition

AIRCRAFT VIBRATION AND FLUTTER

ELEMENTS OF

Mechanical Vibration

C. R. FREBERG
Head, Engineering Research Division
Southern Research Institute

E. N. KEMLER
Associate Director
Southwest Research Institute

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PREFACE

The increased emphasis which engineers must give to noise elimination has indicated the desirability of including a fundamental discussion of the engineering aspects of this subject. Accordingly, a chapter on sound and its engineering applications has been included in the second edition of this book. Since beams are a very common structural element, they have been given additional emphasis by the previous material being combined with new material and presented as a separate chapter.

A number of changes and some additions have been made to clarify presentation and to correct errors. The many suggestions made by users of the book have been carefully considered and incorporated into this revision whenever possible. The new material has been included in later chapters so that those who have found the previous edition adaptable to their methods of presentation will need to make only minor rearrangements of their course material.

We are grateful to our many colleagues and students who have used the first edition and have made suggestions and pointed out errors. We hope they will continue to make suggestions in the future.

C. R. FREBERG
E. N. KEMLER

February, 1949

PREFACE TO FIRST EDITION

It is the purpose of this book to discuss in detail the more elementary phases of vibrations and reduce them to a form in which they can be applied to practical problems. Although there are many problems in the field of vibration which can be solved only by using the methods of advanced mathematics, there are also many problems which can be solved by using the simpler forms of differential equations, the approximate methods developed, or the mobility method. These methods of solution can be mastered by engineers and students without training in advanced mathematics. The more common problems requiring only these simple methods include isolation of equipment and determination of natural frequencies of a great many different types of systems.

The text material has been kept as simple as possible and examples have been included to illustrate the methods, units, and application of the formulas developed. Problems from those involving direct substitution to those requiring difficult derivations have been included to permit the instructor to adapt the course to the degree of difficulty desired.

The mobility method has been introduced. The method offers a relatively simple means of determining the behavior of systems of several degrees of freedom. Systems with non-linear springs, self-induced vibrations, solid friction, etc., have not been considered. These and other less frequently used topics are discussed in the literature and in several of the treatises on vibrations.

The student should familiarize himself with such excellent treatises as those by Timoshenko,[1]* Den Hartog,[2] Wilson,[3] and Kimball.[4] The bibliography gives many references to some of the useful articles on vibration. The authors have undoubtedly been influenced by the above treatises and by the many excellent articles on the subject. Specific acknowledgments have been made where the material has been conscientiously used.

The authors are grateful to their colleagues and students who have made suggestions during the preparation of this material. They particularly wish to acknowledge the helpful suggestions made by Dean A. A. Potter and Mr. Nicholas Kulik of Purdue University, Professor E. L. Midgette of the Polytechnic Institute of Brooklyn, and the

* These numbers refer to the bibliography at the end of the book.

reviewers of the manuscript. They are indebted to Mr. G. F. Nordenholt, editor of *Product Engineering*, for permission to use some of the material appearing in Chapters VI, VII, and VIII. Acknowledgment has been made in the text to manufacturers for illustrations of vibration isolation and engineering data. The authors are grateful for the assistance given by Virginia D. Freberg and Doris M. Kemler during the preparation of the manuscript and handling of the proof.

C. R. FREBERG
E. N. KEMLER

June, 1943

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SYMBOLS FOR PHYSICAL QUANTITIES

A	Area, a constant
a	Acceleration in in./sec^2
B	A constant
C	Constant
$\mathbf{C}$	Capacitance in farads
c	Distance from neutral axis to extreme fiber in inches
D	Constant
d	Diameter in inches
db	Decibels
E	Modulus of elasticity in lb/in.^{2}
$\mathbf{E}$	Voltage in volts
e	Natural logarithmic base = 2.7183
F	Force in pounds
f	Frequency of harmonic motion
G	Shear modulus of elasticity lb/in.^{2}
g	Acceleration due to gravity = 386 in./sec^2
I	Mass moment of inertia in in.-lb-sec^2
I	Sectional moment of inertia in in.^{4}
i	$\sqrt{-1}$
$\mathbf{I}$	Current in amperes
J	Polar moment of inertia in in.^{4}
KE	Kinetic energy in in.-lb
k	Linear spring constant in lb/in.
k_t	Torsional spring constant in lb/in.
L	Length
$\mathbf{L}$	Inductance in henrys
M	Moment of force, bending moment in in.-lb
m	Mass in lb-sec^2/in.
N	Revolutions per minute (rpm)
n	Any number
P	Force or load in pounds
PE	Potential energy in in.-lb
p	Pressure per unit area in lb/in.^{2}
$\mathbf{Q}$	Charge in coulombs
R	Radius in inches
$\mathbf{R}$	Resistance in ohms
r	Linear damping constant in lb-sec/in.
r_t	Torsional damping constant in in.-lb-sec/radian
s	Stress in lb/in.^{2}
S	Surface area
α	Sound absorption coefficient
τ	Sound transmission coefficient

CHAPTER I

INTRODUCTION

1·1. Engineering Applications of Vibrations. Vibration problems occur in every branch of engineering. Wherever there is moving or rotating machinery, vibrations as a result of the unbalanced forces may be induced in parts of the machinery itself or in adjoining equipment or structures. Many types of vibration occurring in nature also affect engineering structures or equipment. Earthquakes, for example, cause vibrations affecting buildings primarily, although many other engineering structures such as dams or pipe lines may be damaged. Ocean waves affecting the motion of ships are undesirable because of their effect on human comfort. When they become large they also endanger the safety and efficiency of the ship. Wind may produce vibrations which damage such engineering structures as icy transmission and telephone lines [5]* or suspension bridges.[6]

Vibrations may result in the failure of machinery parts. Examples of such failure can be found in crankshafts of automobiles or airplane engines,[7] propeller shafts in boats,[8] wing flutter in airplanes,[9,34] and front wheel shimmy in automobiles.[10] Vibrations cause loss of efficiency and may also result in objectionable noise or motion. Some examples are vibrations in engine manifolds which may affect the performance of an engine,[11] shaking of floors or supports, noise from the exhaust of engines, or rough riding of automobiles because of improper support. Continued vibrations in machinery may cause excessive wear and loosening of parts to a degree that will interfere with the proper operation of the equipment. Delicate instruments must be protected from vibrations if they are to operate satisfactorily.

Such objectionable vibrations, if properly analyzed, can frequently be eliminated, or the resultant vibration can be reduced to the point where it is no longer objectionable. Vibrating machinery can be isolated, supports to transmit a minimum of energy can be installed, shock absorbers can be installed, mufflers or screens can be used to eliminate noise, or the cause of the troublesome vibration may in a few cases be eliminated by proper analysis.

* These numbers refer to the bibliography at the end of the book.

Frequently vibrations are desirable so that as much time may be spent in getting the required vibration as in eliminating an undesirable vibration. Every musical instrument involves vibration production; shakers for sorting or screening materials require vibrations; electric clippers or shavers use a vibrating system to actuate the cutting blade;[12] the vibration of a cantilever beam is used as the basis of determining speeds in the case of the Frahm tachometer; exercising machines frequently employ some vibration device; and some types of fatigue and endurance testing equipment use a vibrating system for producing the motion.[13]

1·2. Important Definitions. When undertaking the study of a new subject, the student is confronted with new terminology. The understanding of the material in the text is facilitated if the meaning of these new terms is made known early. The following list of definitions gives the meaning of some of the new or frequently used terms which will be introduced.

Vibration. A body, machine element, or machine is said to vibrate when it executes a periodic motion about a position of equilibrium.

Periodic Motion. A periodic motion is one which repeats itself at definite intervals of time.

Simple Harmonic Motion. Simple harmonic motion is the simplest type of vibratory motion. It can be defined as the projection of point P on the diameter (point Q in Fig. 1·1) as P moves around the circle with uniform angular velocity. The distance of Q from the midpoint of its path is shown as $x = OQ$. It can be written that

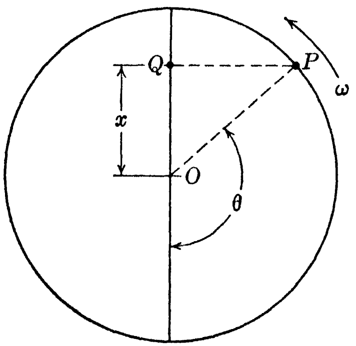


FIG. 1·1.

$$x = OQ = r\cos\theta$$

If the point P is rotating about the point O at some constant angular velocity ω, the angle θ at any time t is

$$\theta = \omega t$$

so that we can write

$$x = r\cos\omega t$$

The velocity of the projection, obtained by taking the derivative of the displacement, is

$$v = \frac{dx}{dt} = -r\omega\sin\omega t$$

Likewise the acceleration can be found as

$$a = \frac{dv}{dt} = -r\omega^2\cos\omega t$$

If it is desirable, x may be substituted for $(r \cos \omega t)$ in this expression to give

$$a = -x\omega^2$$

The acceleration is then seen to be proportional to the distance x that the point Q is displaced from the center of its path. Simple harmonic motion is frequently defined as motion where the acceleration is proportional to the displacement from the midpoint of the motion and is always directed toward that point.

Frequency. The frequency of a motion is the number of times the motion repeats itself per unit of time.

Free Vibration. Free vibration is that periodic motion which takes place when an elastic system is displaced from its equilibrium position and released. In this case the forces acting on the system after its release are dependent only on the motion of the system.

Forced Vibrations. When the vibration results from the application of an external periodic force it is called a forced vibration.

Transient Conditions. Whenever the forces acting upon a dynamic system change, there is a change in the motion of that system. Any phenomena which occur during the time required for the system to adapt itself from one force system to another are called transient phenomena.

Steady State Conditions. After any vibrating system has been acted upon by a definite force system for a sufficient time, it will follow a definite cycle of events. When such a time is reached, the system is said to be in a steady state condition.

Natural Frequency. The frequency of a free vibration is called the natural frequency of the system.

Resonance. When a system is acted upon by an external periodic force having the same frequency as the natural frequency of the system, the amplitude of the system will become very large, and the system is said to be in a state of resonance.

Critical Speed. A critical speed exists when the frequency of the disturbing force equals or approaches the natural frequency.

Period. The time required for the motion to complete one cycle is called the period. (See Fig. 1·2.) This is the reciprocal of the frequency.

Amplitude. The amplitude is defined as the distance from the mean position to the point of maximum displacement, as indicated in Fig. 1·2.

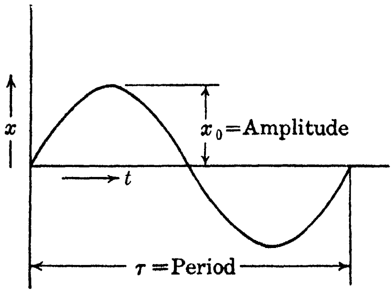


FIG. 1·2.

Phase Angle. The angle between two rotating vectors or two harmonic waves of the same frequency is known as the phase angle. The displacement, velocity, and acceleration in the case of simple harmonic

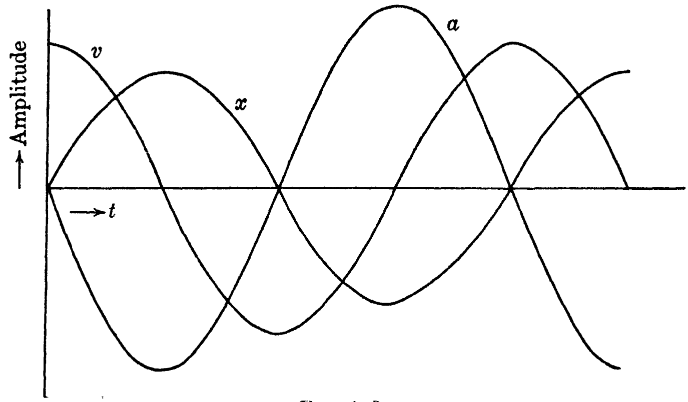


FIG. 1·3.

motion are functions of sine or cosine and therefore are harmonic functions. They repeat themselves with a frequency which represents the number of times the system goes through all phases of its motion. It will be noted from Fig. 1·3 that the displacement, velocity, and acceleration all have the same shape curve except that they are displaced with

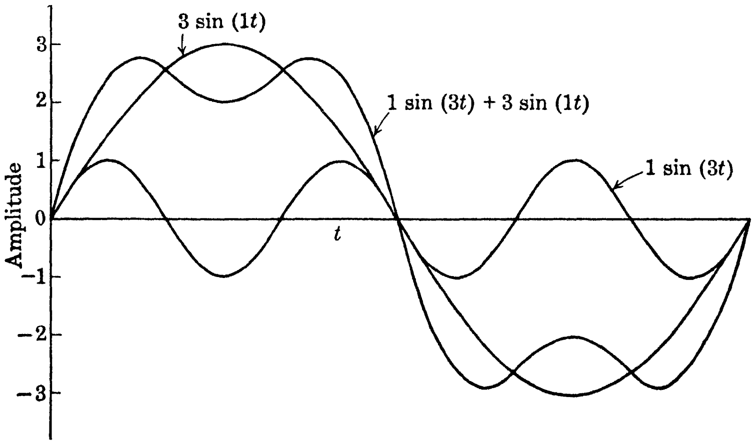


FIG. 1·4.

respect to time or angle. The velocity curve leads the displacement curve by 90° or $\frac{1}{4}$ cycle, and the acceleration leads the displacement by 180° or $\frac{1}{2}$ cycle. This angular displacement between any two harmonic waves of the same frequency is the phase angle.

Harmonics. Any periodic motion which is not simple harmonic can be considered as being a sum of simple harmonic motions of frequencies which are multiples of the frequency of the motion. In general any periodic motion can be expressed in the form

$$x = A_0 + A_1 \sin \omega t + A_2 \sin 2\omega t + \cdots$$
$$+ B_1 \cos \omega t + B_2 \cos 2\omega t + \cdots$$

Figure 1·4 shows a simple example.

Rotating Vectors. The motion of point P in Fig. 1·1 can be represented by a rotating vector drawn from point O. A harmonic wave will

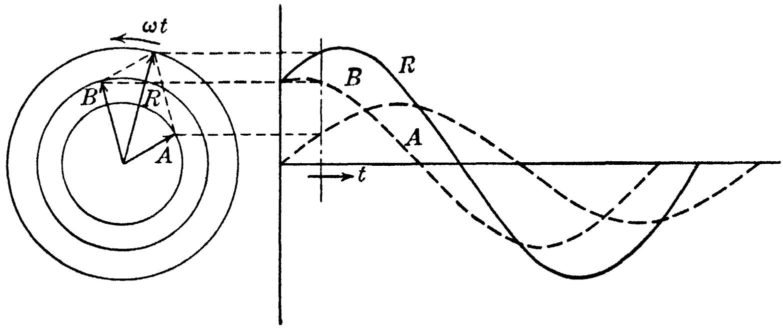


FIG. 1·5.

be the projection of this vector on the diameter. The motion resulting from the combination of several simple harmonic motions of the same frequency can be represented by the sum of the vectors of the individual motions as indicated in Fig. 1·5.

Mobility. Mobility can be defined as the ease of motion, or, more specifically, the velocity which will result from the application of a unit force.

Damping. When friction is present in a system the resulting vibration of the system is called a damped vibration.

Aperiodic Motion. When the motion takes place in one direction only it is called aperiodic motion. This type of motion is encountered when critical damping is present.

Statics. Statics is that part of mechanics dealing with bodies at rest. It involves finding the conditions under which a system of forces will place the body upon which they act in equilibrium. These conditions are fulfilled when two requirements are satisfied, namely,

$$\Sigma F = 0$$
$$\Sigma M = 0$$

Kinematics. Kinematics is that portion of mechanics dealing with the motion of bodies without reference to the forces acting on the body.

Dynamics. Dynamics is that portion of mechanics that concerns the forces set up by reason of a body's motion. A body that is acted upon by an unbalanced applied force system will be given an acceleration proportional to the unbalanced force and in the direction of the unbalanced force. The solution of problems of dynamics can be made either by summing up all the applied forces and equating them to the resulting inertia force $\frac{W}{g}a$, or by placing on the system the so-called reversed effective force $\frac{W}{g}a$ in a direction opposite to the acceleration, and then using the methods which apply to systems in equilibrium. In the following discussion the former relation for dynamic systems is used. This can be written as

$$\Sigma F = \frac{W}{g}a$$

For rotating systems an equivalent relation exists which can be written as

$$\Sigma T = I\alpha$$

1·3. Symbols and Units. The application of the theory to practical problems requires that care be given to the use of units and symbols. Some of the symbols which have been more commonly used in the text, together with the units used, are listed in the front of the book.

The inch-pound-second system is used in vibration work because of the small displacements normally involved. For example, the acceleration of gravity, g, is expressed as 386 in./sec^2 in vibration problems instead of the conventional 32.2 ft/sec.^{2} These changes in units become very important when practical problems are to be solved. Extreme care must be taken in the substitution of values to see that the proper units are used.

CHAPTER II

VIBRATIONS WITHOUT DAMPING

2·1. Introduction. Many engineering vibration problems can be represented by a simple system consisting of an elastic member such as a spring and a concentrated weight or mass. Such systems require only one coordinate to describe them, and are called systems of one degree of freedom. A characteristic of such systems, which is of particular engineering interest, is the natural frequency. The natural frequency is the rate at which a system will vibrate if displaced from its equilibrium position and released. Since it can be determined easily and often supplies enough information for solving or understanding many engineering vibration problems, it deserves considerable attention.

Systems of one degree of freedom can be subdivided in several ways. One method is to divide them according to the source of energy or force causing a vibration. Such a classification would divide the field into free and forced vibrations. A free vibration is one which takes place when a system is displaced from its equilibrium position and released with no further force or energy being supplied. The spring and weight shown in Fig. 2·1 comprise a typical vibration system.

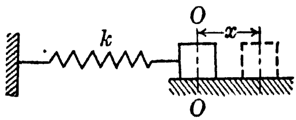


FIG. 2·1.

If the weight is displaced from its equilibrium position as indicated, the spring will be placed in tension. If the weight is released, the spring force will tend to return the weight to its equilibrium position. The energy given up by the spring (work done by the spring as it contracts) will cause the weight to speed up. (Assume friction to be neglected.) When the weight reaches its equilibrium position (unstretched position of spring) there will be no force exerted by the spring. Because of its velocity the weight will keep moving. As the weight moves past the equilibrium position the spring will be compressed. The energy required to compress the spring or the force exerted by the spring will cause the velocity of the weight to decrease. When the spring has reached a position such that it has absorbed all the energy in the moving weight, the weight will be at rest. The compressed spring will then cause the weight to move in the reverse direction. As the weight moves back and forth there will be a

transfer of energy between the weight and spring. This interchange of energy is present in all free vibrations and, as will be seen, can be used as the basis for making free vibration calculations. Forced vibrations are those which take place when the vibrating system has some external periodic force such as an unbalanced shaking force applied to it.

Vibration systems may also be divided into ideal systems where there is no friction or energy loss and into actual systems where there is always some energy dissipation or damping. The ideal systems are much easier to set up and solve. Since many actual systems have very little damping in them, a good approximation of the critical frequency may be made by neglecting the damping. This chapter considers only ideal systems where no damping is present. It also will be limited to systems of a single degree of freedom. The effect of damping or friction will be considered in the following chapter.

FREE VIBRATIONS OF SPRING AND WEIGHT WITHOUT DAMPING

2·2. Derivation of the Equation of Motion. One fundamental and typical type of vibration problem encountered in engineering work can be represented by the frictionless spring and weight system shown in Fig. 2·1. As indicated previously, such a system will vibrate if it is displaced from its equilibrium position and released. This can be verified mathematically by applying Newton's equations of motion.

In most types of vibrating systems there is an elastic member such as a spring in which potential energy may be stored. In the type of elastic members considered here, the force exerted by such members is assumed to be proportional to the deformation or deflection of such members. The load-deflection diagram of a spring, beam, or any other elastic member can be represented graphically by a straight line, as indicated in Fig. 2·2. The relation between this force and the displacement is expressed analytically as $F = kx$, where x is the displacement and k is the proportionality constant, more commonly known as the spring constant. The equation of motion for the system shown in Fig. 2·1 can be determined by applying the condition of equilibrium

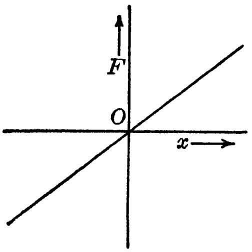


FIG. 2·2.

$$\Sigma F = \frac{W}{g} a = \frac{W}{g} \frac{d^2x}{dt^2}$$

which states that the sum of the external forces applied to a body is

equal to the product of the mass and the acceleration and that the force acts in the direction of the acceleration. Therefore, the spring force that is applied to the weight acts in the same direction as the acceleration of the weight. Since the force is proportional to the displacement, the acceleration is proportional to the displacement and is directed toward the equilibrium position. This indicates immediately that the motion is simple harmonic so that the displacement can be written as

$$x = x_0 \sin \omega t$$

or

$$x = x_0 \cos \omega t$$

If, therefore, the motion were plotted against time, the curve indicating displacement would be a sine or cosine wave. It was shown in Chapter I that the projection of a vector rotating at constant angular velocity moves with simple harmonic motion. The angular position of this vector is given by the term ωt, where ω is the angular velocity of the rotating vector. This indicates that one wave or cycle is completed when

$$\omega t = 2\pi \text{ radians}$$

The number of cycles per second or natural frequency will be

$$f = \frac{\omega}{2\pi}$$

To find an expression for the frequency it is necessary to set up a force equation. Using Newton's law, we obtain the following relation:

$$\Sigma F = -kx = \frac{W}{g}\frac{d^2x}{dt^2} \qquad [2\cdot 1]$$

The negative sign indicates that the displacement is opposite the acceleration. A general method for determining the direction of the signs in this equation is somewhat more involved.*

* In vibration problems forces are expressed in terms of the displacement and its derivatives. Since both the forces and the displacement or its derivatives are vector quantities both must have proper signs assigned to them. When they are expressed in analytical form, it may be difficult to decide whether a sign must be assigned or is inherent in the term. If a positive sign is assigned to the displacement when it is to the right, for example, the velocity and acceleration will automatically be positive when they are in that direction. It is also necessary to designate a sign for the forces which will be taken as positive in the same direction as positive displacement. A spring force will then always be opposite the displacement and hence will be written as $-kx$. A friction or damping force proportional to the velocity and opposed to the

Equation 2·1 may also be written

$$\frac{W}{g}\frac{d^2x}{dt^2} + kx = 0$$

or

$$\frac{d^2x}{dt^2} + \frac{kg}{W}x = 0 \qquad [2\cdot 2]$$

These equations represent the equation of motion for the spring and weight system expressed in differential form. The next step in the problem is to obtain a solution of this equation: that is, to find an expression which will give the displacement of the weight from its equilibrium position as a function of time and the physical properties of the system.

2·3. Solution of the Differential Equation. The generalized solution of vibration problems requires the solving of differential expressions of the form of equation 2·2, called differential equations. The simplest forms of differential equations are such expressions as $\frac{dy}{dx} = 2$, $\frac{dy}{dx} = x$, $\frac{dy}{dx} = f(x)$. These equations can be solved by direct integration according to the methods developed in any book on elementary calculus. A more common type of differential equation occurring in vibration work has the general form

$$A_0\frac{d^2x}{dt^2} + A_1\frac{dx}{dt} + A_2x = 0 \qquad [2\cdot 3]$$

The form of this equation is entirely different from those which can be solved by direct integration.

The problem of finding the differential form of an equation corresponding to a particular mathematical expression is direct and involves only differentiation and elimination of constants. The inverse problem of finding the solution of a differential equation is not direct and cannot be determined by any one general method. Any mathematical relation which will satisfy the differential equation is a solution.

The solution of an equation such as 2·3 requires some function which, when continually differentiated, will not change form. Obviously an expression of the form

$$x = Ce^{mt} \qquad [2\cdot 4]$$

---

velocity will always be written $-r\frac{dx}{dt}$; and, since the inertia force is porportional to the acceleration and in the direction of the acceleration, it will always be written $+\frac{W}{g}\frac{d^2x}{dt^2}$. As x, $\frac{dx}{dt}$, or $\frac{d^2x}{dt^2}$ change from + to −, the sign of the force terms will automatically change.

would satisfy this requirement. If this expression is differentiated, the values are

$$\frac{dx}{dt} = Cme^{mt}; \qquad \frac{d^2x}{dt^2} = Cm^2e^{mt}$$

Substitution of these values in equation 2·3 would give

$$A_0Cm^2e^{mt} + A_1Cme^{mt} + A_2Ce^{mt} = 0$$

or

$$A_0m^2 + A_1m + A_2 = 0 \qquad [2\cdot5]$$

Equation 2·4 will be a solution of equation 2·3 provided m is selected so that equation 2·5 becomes zero. If m is a root of the equation this condition will be satisfied. It can be shown that either of the two roots for m as obtained from equation 2·5 would be a solution of the equation. The sum of these two solutions would give the general solution of the differential equation.

A second order differential equation with constant coefficients of the general form of equation 2·3 may then be solved by following a definite procedure. An auxiliary algebraic equation of the form of equation 2·5 may be written as

$$A_0m^2 + A_1m + A_2 = 0 \qquad [2\cdot5]$$

Solving for the roots of this algebraic equation, we obtain

$$m_1 = \frac{-A_1 + \sqrt{A_1^2 - 4A_0A_2}}{2A_0}, \quad m_2 = \frac{-A_1 - \sqrt{A_1^2 - 4A_0A_2}}{2A_0} \qquad [2\cdot6]$$

If m_1 and m_2 are real, the general solution can be written in the convenient form

$$x = C_1e^{m_1t} + C_2e^{m_2t} \qquad [2\cdot7]$$

If m_1 and m_2 are complex, the roots given by equation 2·6 take the form

$$m_1 = a + bi \qquad [2\cdot8]$$

$$m_2 = a - bi$$

where $i = \sqrt{-1}$.

The solution for the case where the roots are complex can be written in the form of equation 2·7 if desired, but this form of the solution is not easily applied. It is more useful and easier to write it in the equivalent trigonometric form. (See any reference on differential equations or complex algebra.)

$$x = e^{at}(C_1 \cos bt + C_2 \sin bt) \qquad [2\cdot9]$$

The steps in the procedure outlined may now be followed to obtain a solution to equation 2·2. The auxiliary equation can be written as

$$m^2 + \frac{kg}{W} = 0 \tag{2·10}$$

The roots of this auxiliary equation are

$$m_1 = 0 - i\sqrt{\frac{kg}{W}}$$

$$m_2 = 0 + i\sqrt{\frac{kg}{W}}$$

The solution can now be written in exponential form, but it is more convenient and useful to write the solution in the trigonometric form of equation 2·9. Thus the general solution of equation 2·2 is

$$x = A \sin \sqrt{\frac{kg}{W}}\, t + B \cos \sqrt{\frac{kg}{W}}\, t \tag{2·11}$$

The quantities A and B given in this equation are arbitrary constants which take into account the initial conditions under which the vibration starts. Their evaluation will be discussed later.

2·4. Alternative Method of Solving the Differential Equation. A simple type of differential equation such as equation 2·2 can often be solved by inspection and the solution verified by substitution. By examining equation 2·2, it may be found that the solution must be some type of function which when differentiated twice will have the same form. This is necessary in order that it may be eliminated upon substitution in the differential equation. There are two types of functions that will satisfy such conditions. They are exponential and trigonometric functions. For example, $x = A \sin \omega t$ would be one solution provided the proper value of ω is selected. This can be verified by differentiation and substitution. Differentiation gives

$$\frac{dx}{dt} = +A\omega \cos \omega t$$

$$\frac{d^2x}{dt^2} = -A\omega^2 \sin \omega t$$

If these values are substituted in equation 2·2, the result is

$$-A\omega^2 \sin \omega t + \frac{kg}{W} A \sin \omega t = 0$$

When this is divided by $-A \sin \omega t$, the result is

$$\omega^2 - \frac{kg}{W} = 0$$

or

$$\omega = \sqrt{\frac{kg}{W}} \qquad [2\cdot 12]$$

One solution would then be

$$x = A \sin \sqrt{\frac{kg}{W}}\, t \qquad [2\cdot 13]$$

It can be seen that

$$x = B \cos \sqrt{\frac{kg}{W}}\, t \qquad [2\cdot 14]$$

is also a solution.

From mathematical considerations it can be shown that there must be two arbitrary constants to have a general solution to a second order equation such as equation 2·2. This means that the solution, in its most general form, must have two arbitrary constants. Further, it can be shown that the sum of the particular solutions given by equations 2·13 and 2·14 is also a solution and will be the general solution. Therefore, the general solution is

$$x = A \sin \sqrt{\frac{kg}{W}}\, t + B \cos \sqrt{\frac{kg}{W}}\, t \qquad [2\cdot 11]$$

As suggested previously, the solution may be put in exponential form. It can be found by trial, just as in the case of trigonometric functions, that

$$x = Ae^{\sqrt{-\frac{kg}{W}}\, t}$$

is a solution of equation 2·2. The general solution when written in this form is

$$x = Ae^{\sqrt{-\frac{kg}{W}}\, t} + Be^{-\sqrt{-\frac{kg}{W}}\, t} \qquad [2\cdot 15]$$

where A and B are again arbitrary constants depending upon initial conditions of motion.

2·5. Evaluation of Arbitrary Constants. In the general solutions of the differential equations, A and B are arbitrary constants whose values depend on the initial conditions. One set of conditions commonly used is that where the weight is displaced a distance x_0 from its equilibrium

position and then released without initial velocity. These conditions can be expressed analytically as

$$x = x_0 \text{ when } t = 0 \tag{2·16}$$

and

$$\frac{dx}{dt} = 0 \text{ when } t = 0 \tag{2·17}$$

The first of these conditions indicates that the body is displaced a distance x_0 when we begin to measure time. The second condition indicates that it has zero velocity when it is released. By applying these conditions to equation 2·11, we find that

$$x_0 = A(0) + B(1)$$

or

$$B = x_0$$

The condition given by equation 2·17 yields

$$\frac{dx}{dt} = 0 = A\sqrt{\frac{kg}{W}}\cos\sqrt{\frac{kg}{W}}\,(0) - x_0\sqrt{\frac{kg}{W}}\sin\sqrt{\frac{kg}{W}}\,(0)$$

This gives

$$A = 0$$

The complete solution for the motion of the weight W shown in Fig. 2·1 for the initial conditions given by relations in equations 2·16 and 2·17 is

$$x = x_0 \cos\sqrt{\frac{kg}{W}}\,t \tag{2·18}$$

Because of the initial conditions assumed, equation 2·18 becomes a particular solution of equation 2·2. It can be used to determine the natural frequency and displacement at any time t. It shows that the motion repeats itself periodically, and that under ideal conditions the motion will not diminish as time goes on. It is known from experience, however, that eventually it must stop because of air resistance and internal friction. Nevertheless, in many cases the reduction is very slow so that it may be used for calculating the natural frequency with good engineering accuracy.

For a case where the weight has an initial velocity v_1 when displaced a distance x_1, the initial conditions are

$$x = x_1 \text{ when } t = 0$$

$$\frac{dx}{dt} = v_1 \text{ when } t = 0$$

The same method may be followed in evaluating the constants as before to obtain the result

$$x = \frac{v_1}{\sqrt{\frac{kg}{W}}} \sin \sqrt{\frac{kg}{W}}\, t + x_1 \cos \sqrt{\frac{kg}{W}}\, t \qquad [2 \cdot 19]$$

Illustrative Problem. In Fig. 2·1 consider the case where a weight of 2 lb is attached to a spring having a spring constant $k = 20$ lb/in. If $x_0 = 0.1$ in., we have by substituting in equation 2·18

$$x = 0.1 \cos \sqrt{\frac{20(386)}{2}}\, t = 0.1 \cos (62.1)t$$

The displacement of the weight from its equilibrium position at any time, t, can be determined by substituting values of t into the above equation. The velocity of the weight can be found by differentiating the equation for displacement, giving

$$v = \frac{dx}{dt} = -6.21 \sin (62.1)t$$

Again by substituting values of t in this equation the velocity at time t may be determined. The acceleration can be found in like manner by differentiating the velocity equation, giving

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -386 \cos (62.1)t$$

If desirable the value of $0.1 \cos (62.1)t$ from the displacement equation can be substituted in this equation, giving

$$a = \frac{d^2x}{dt^2} = -3860x$$

The acceleration can then be found in terms of the displacement or in terms of time, depending on the known conditions.

The equation of displacement, in addition to giving information on the displacement of the weight, can be used to determine forces and therefore stresses in the spring. This case is so simple as to be obvious, but this method of determining loads and stresses can be applied to more complex cases. The force in this system, which is equal to the displacement times the spring constant k, can be expressed as

$$F = kx = k(0.1) \cos (62.1)t$$

The maximum stress can be determined by substituting this force in the equation for spring stress.

2·6. Equivalent Vector Motion. In examining the solution as given by equation 2·11, we see that the solution is composed of the sum of two simple harmonic terms. Since a simple harmonic motion may be represented by the projection of a rotating vector on a diameter, the sum of a sine term and a cosine term may be obtained by adding the vectors as shown in Fig. 2·3. A cosine term is always 90° ahead of a sine term so that the vectors representing the two terms in equation 2·11 are at right angles. The resultant motion will be the same as the resultant vector. The resultant motion can be represented by the equation

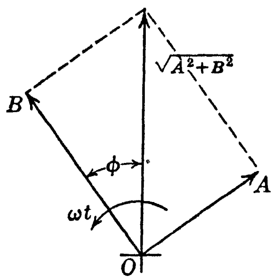


FIG. 2·3.

$$x = \sqrt{A^2 + B^2} \cos\left(\sqrt{\frac{kg}{W}}\, t - \phi\right) \qquad [2\cdot 20]$$

where ϕ = phase angle = $\tan^{-1} \dfrac{A}{B}$.

The term $\sqrt{\frac{kg}{W}}\, t$ in any of the above equations represents the angle turned through by the rotating vectors. Since the angle is the product of the angular velocity and the time, the term $\sqrt{\frac{kg}{W}}$ corresponds to the equivalent angular velocity of simple harmonic motion or

$$\omega_n = \sqrt{\frac{kg}{W}} \text{ radians/sec}$$

Physically ω_n has no direct meaning, but, since the motion completes one whole cycle while the vector is turning 2π radians, the number of cycles occurring per second is

$$f = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{kg}{W}} \text{ cycles/sec} \qquad [2\cdot 21]$$

This is known as the natural frequency of the system because no external force is needed to maintain the vibration at this frequency.

2·7. Energy Method of Solving Spring and Weight Problems. Problems involving systems similar to the simple spring and weight problem shown in Fig. 2·1 can often be worked to advantage by using energy methods. Natural vibrations of systems without damping or friction involve the interchange of kinetic energy stored in the weight due to its motion and potential energy stored in the elastic members, such as the

spring. The energy stored in the spring at any displacement x can be determined graphically from Fig. 2·2 or by using the general principle that work equals $\int F dx$. Since $F = kx$, the potential energy or work, PE, stored in the spring will be

$$PE = \int F dx = \int_0^x (kx) dx = \tfrac{1}{2} kx^2 \qquad [2\cdot 22]$$

This same value can be determined from Fig. 2·2 by noting that the work done by any variable force is equal to the area under the force-displacement curve. This area or work will then be one-half the product of the height of the triangle, kx, and the base of the triangle at the displacement x, or $\frac{1}{2}(x)(kx) = \frac{1}{2}kx^2$, which is the same value as given by equation 2·22.

The kinetic energy due to the motion of the system can be calculated from the formula

$$KE = \frac{1}{2} \frac{W}{g} (\text{velocity})^2$$

Since the velocity in this case is not constant, it must be expressed analytically as

$$\text{Velocity} = \frac{dx}{dt}$$

giving

$$KE = \frac{1}{2} \frac{W}{g} \left(\frac{dx}{dt} \right)^2 \qquad [2\cdot 23]$$

Since there is no dissipation of energy, the energy possessed by the system in any position is constant and equal to the sum of the potential energy and the kinetic energy,

$$PE + KE = C$$

If the derivative of this equation is taken with respect to time, the following expression is obtained:

$$\frac{1}{2}(2kx)\frac{dx}{dt} + \frac{1}{2}\left(2\frac{W}{g}\right)\frac{d^2x}{dt^2}\frac{dx}{dt} = 0 \qquad [2\cdot 24]$$

Equation 2·24 gives the conditions under which the rate of change of energy in the system with respect to time is zero. This is a mathematical way of stating that the energy in the system will be constant. Upon

simplification this equation reduces to the following form, which is identical with equation 2·2:

$$\frac{d^2x}{dt^2} + \frac{kg}{W}x = 0$$

The solution is therefore given by equation 2·11.

Illustrative Problem. An electric motor weighing 100 lb is mounted on 4 springs, each of which has a spring constant of 1000 lb/in. The motor is guided so that it can move only in a vertical direction. Determine the natural frequency of this system.

Solution. In this case the springs are in parallel, giving the effect of a spring having the stiffness of four times the individual springs. The effective k will then be 4000 lb/in. The weight carried on the springs is 100 lb. From equation 2·21 the frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{4000(386)}{100}} = 19.8 \text{ cycles/sec}$$

$$= 19.8(60) = 1188 \text{ cycles/min}$$

This frequency is very near the speed of a 1200 rpm motor. If such a motor were used, trouble from the vibration would undoubtedly result.

2·8. Weight Suspended on a Vertical Spring. Figure 2·4 shows a weight suspended from a spring. If x_1 represents the displacement from the position where there is zero stress in the spring and δ_{st} is the static displacement of the weight, the equation of motion will be

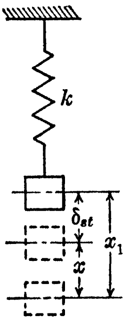


FIG. 2·4.

$$\frac{W}{g}\left(\frac{d^2x_1}{dt^2}\right) + kx_1 - W = 0 \qquad [2\cdot25]$$

Then let

$$x = x_1 - \delta_{st} = x_1 - \frac{W}{k} \qquad [2\cdot26]$$

which will give

$$\frac{d^2x}{dt^2} = \frac{d^2x_1}{dt^2}$$

Equation 2·25 will then reduce to

$$\frac{W}{g}\frac{d^2x}{dt^2} + kx = 0$$

The solution of this equation is obviously the same as that of the case shown in Fig. 2·1. It should be noted that

$$\delta_{st} = \frac{W}{k}$$

and the frequency can be given by

$$f = \frac{60}{2\pi}\sqrt{\frac{kg}{W}} = \frac{60}{2\pi}\sqrt{\frac{g}{\delta_{st}}} = 187.7\sqrt{\frac{1}{\delta_{st}}} \text{ cycles/min} \qquad [2\cdot 27]$$

Therefore, δ_{st}, the static deflection, can be used to determine the natural frequency of the system.

EXAMPLES OF FREE VIBRATION WITHOUT DAMPING

2·9. U Tube. The equations as given above cover specifically the case of a weight acting on a spring. The principles developed actually cover any problem where the restoring force is proportional to the displacement. In Fig. 2·5 the restoring force is the unbalance weight in the manometer. The weight being accelerated is the weight of the liquid in the manometer. The mass is

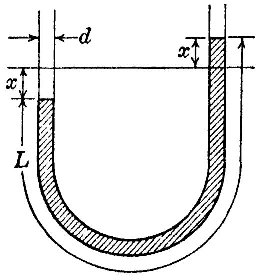


Fig. 2·5.

$$\frac{W}{g} = \frac{\pi d^2 L}{4}\frac{w}{g}$$

where w is the specific weight of the fluid and W is the total weight of the fluid in the manometer. The restoring force, due to position, is

$$F = \frac{w\pi d^2}{4}(2x)$$

From Newton's law it is then possible to write

$$\frac{\pi d^2 L}{4}\left(\frac{w}{g}\right)\frac{d^2x}{dt^2} = -\frac{w\pi d^2}{4}(2x)$$

or

$$\frac{d^2x}{dt^2} + \frac{2g}{L}x = 0 \qquad [2\cdot 28]$$

It is obvious that the term $2g/L$ corresponds to kg/W in the problem of the mass and spring. This means that the displacement would be given by

$$x = x_0 \cos \sqrt{\frac{2g}{L}}\, t \qquad [2\cdot 29]$$

and the natural frequency by

$$f = \frac{1}{2\pi}\sqrt{\frac{2g}{L}} \text{ cycles/sec} \qquad [2\cdot 30]$$

Illustrative Problem. A U tube of $\frac{1}{4}$ in. inside diameter has a mercury column which is 8 in. long. This U tube is to be used for measuring the pressure on the discharge of a low-pressure single-cylinder blower which makes 100 working strokes/min. Is there any danger of the natural frequency of the gage being near the frequency of the air pulsations from the blower?

Solution. From equation 2·30 we see that the frequency is independent of the fluid in the manometer and is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{2(386)}{8}} = 1.56 \text{ cycles/sec}$$

$$= 98 \text{ cycles/min}$$

Since the natural frequency and the frequency of the compressor discharges are so close, it is probable that difficulty might be experienced in getting the fluid in the gage to stand still.

This difficulty could be avoided by using a longer or shorter length of mercury in the U tube. For example, a 16-in. length would give a natural frequency of 70 cycles/min, which would be well below the critical value.

2·10. Pendulum. Figure 2·6 shows another problem of the same general type. The force tending to restore the weight to its equilibrium position is $W \sin \theta$. If the angle through which the pendulum is displaced is small, then the approximation that $\sin \theta = \theta$ can be made. This can be verified by checking values from a trigonometric table, keeping in mind that θ must be used in radians on the right-hand side of this expression. In Newton's equation the tangential acceleration is

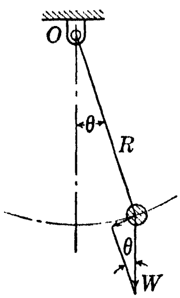


FIG. 2·6.

$$\text{Acceleration} = R\frac{d^2\theta}{dt^2}$$

and the restoring force is

$$F = -W \sin \theta = -W\theta \quad \text{(approx.)}$$

Using these values, we can obtain an equation equivalent to 2·2, giving

$$\frac{W}{g}R\frac{d^2\theta}{dt^2} + W\theta = 0$$

or

$$\frac{d^2\theta}{dt^2} + \frac{g}{R}\theta = 0 \qquad [2\cdot31]$$

The solution for a pendulum released from rest is

$$\theta = \theta_0 \cos\sqrt{\frac{g}{R}}\,t$$

and

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{R}} \text{ cycles/sec} \qquad [2\cdot32]$$

This equation shows that the frequency of a simple pendulum is dependent only upon its length when the amplitude is small.

Illustrative Problem. A clock is to have a pendulum consisting of a light wire with a weight at its end. If the weight of the wire is neglected, what weight must be placed on the bottom of the wire and what length of pendulum must be used if it is to make one complete cycle per second?

Solution. Equation 2·32 shows that the frequency of the pendulum is independent of the weight placed on it. If the weight is doubled, for example, the force available to restore the pendulum is doubled, and the frequency remains the same. The length, however, can be obtained from equation 2·32 and will be

$$R = \frac{g}{(2\pi f)^2}$$

$$R = \frac{386}{(2\pi 1)^2} = 9.7 \text{ in.}$$

2·11. Spring Constants. The solution of vibration problems requires a knowledge of the system's elastic properties. These properties may be expressed as a spring constant. The linear spring constant for a member subjected to a direct load is usually defined as the force to produce unit extension or compression in the member. It is designated by the symbol k, and the units are generally given as pounds per inch for vibration work. For a helical spring, the relation between load and deflection is

$$F = \frac{Gd^4\delta}{8D^3n}$$

and the spring constant for a helical spring is

$$k = \frac{F}{\delta} = \frac{Gd^4}{8D^3n} \qquad [2\cdot33]$$

where G = shear modulus of elasticity in pounds per square inch
d = diameter of wire in inches
D = mean coil diameter in inches
n = number of coils

The spring constant for a member subjected to torque is the number of inch-pounds of torque required to twist the shaft one radian. The torque on a shaft expressed in terms of angle of twist is

$$T = \frac{\pi d^4 G\theta}{32L}$$

The spring constant for a shaft is given by

$$k_t = \frac{T}{\theta} = \frac{\pi d^4 G}{32L} \qquad [2\cdot34]$$

The spring constant for a cantilever beam, expressed in terms of the load at the end of the beam, is the pounds of force required to produce unit deflection or

$$k = \frac{F}{\delta} = \frac{3EI}{L^3} \qquad [2\cdot35]$$

where E = modulus of elasticity in tension in pounds per square inch
I = moment of inertia of beam section in inches^4
L = length of beam in inches

The spring constant for any type of beam can be found in a similar manner. For a simply supported beam with a load in the middle, the deflection is

$$\delta = \frac{FL^3}{48EI}$$

The spring constant is

$$k = \frac{F}{\delta} = \frac{48EI}{L^3} \qquad [2\cdot36]$$

2·12. Disk and Shaft. A simple system composed of a disk attached to one end of a shaft that is fixed at the other end forms the basis for

solving more complex torsional problems. These problems may be solved by using either energy methods or equilibrium conditions. Figure 2·7 shows a shaft of diameter d supporting a disk having a moment of inertia, I, about its axis of rotation. The spring constant for a uniform shaft of diameter d as given by equation 2·34 is

$$k_t = \frac{G}{L}\frac{\pi d^4}{32}$$

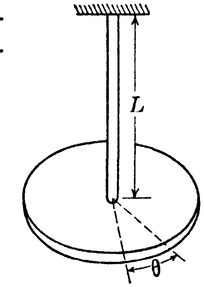


FIG. 2·7.

The torque, T, resulting from a deflection of the disk through an angle θ, is by definition

$$T = -k_t\theta$$

The condition for equilibrium may be determined from the equation

$$\Sigma T = I\alpha$$

which gives

$$-k_t\theta = I\alpha = I\frac{d^2\theta}{dt^2}$$

which can be reduced to the form

$$\frac{d^2\theta}{dt^2} + \frac{k_t}{I}\theta = 0 \qquad [2\cdot37]$$

The solution of this differential equation is obviously the same as for equation 2·2, giving

$$\theta = \theta_0 \cos\sqrt{\frac{k_t}{I}}\,t \qquad [2\cdot38]$$

and

$$f = \frac{1}{2\pi}\sqrt{\frac{k_t}{I}} \text{ cycles/sec} \qquad [2\cdot39]$$

Illustrative Problem. Assume that a propeller drive on a boat can be represented by a shaft and disk combination as shown in Fig. 2·7 because a large flywheel is present on the engine drive shaft. The propeller shaft is 5 in. in diameter and 40 ft long. The moment of inertia of the propeller is 1000 lb-in.-sec^2. What will be the natural frequency of this simplified system?

Solution. From equation 2·34 we find that

$$k_t = \frac{G}{L}\frac{\pi d^4}{32} = \frac{12(10^6)\pi(5^4)}{40(12)32} = 1.53(10^6) \text{ in.-lb/radian}$$

From equation 2·39

$$f = \frac{1}{2\pi}\sqrt{\frac{k_t}{I}} = \frac{1}{2\pi}\sqrt{\frac{1.53(10^6)}{1000}} = 6.24 \text{ cycles/sec}$$
$$= 374 \text{ cycles/min}$$

In cases such as this where the engine flywheel inertia is very large compared with the other values, such an approximation as the above will indicate the lowest speed which would be critical. More accurate analysis of such problems will be given later.

2·13. Torsional Methods for Pendulums. The natural frequency of the pendulum shown in Fig. 2·6 can be worked on the basis of moments as well as on the basis of force, as was done in section 2·10. The moment acting on the pendulum as shown in Fig. 2·6 can be expressed as

$$T = WR\sin\theta$$

If θ is small, $\sin\theta = \theta$, and the torque is approximately equal to

$$T = WR\theta$$

This is the only external torque applied about the pivot point; so the basic equation $\Sigma T = I\alpha$ can be written as

$$-WR\theta = I\frac{d^2\theta}{dt^2}$$

This equation may be rearranged so that it becomes

$$\frac{d^2\theta}{dt^2} + \frac{WR}{I}\theta = 0$$

The frequency of oscillation for any pendulum is

$$\omega_n = \sqrt{\frac{WR}{I}} \qquad [2\cdot40]$$

where I is the moment of inertia about the pivot point or

$$I = I_{cg} + \frac{W}{g}R^2$$

When the mass is small and concentrated the moment of inertia about the center of gravity becomes negligible; so the equation for the frequency becomes

$$\omega_n = \sqrt{\frac{WR}{\frac{WR^2}{g}}} = \sqrt{\frac{g}{R}}$$

This is the same as equation 2·32 given previously for a simple pendulum. Moments of inertia of a body are often determined from the frequency of oscillation for the body about some axis other than one through the center of gravity. Equation 2·40 is used as a basis for determining the total moment of inertia about the pivot point. Any other moment of inertia can be got by the transfer equation also given previously.

A variation of this problem is shown in Fig. 2·8. In this case the torque is given by the sum of the spring force and the gravity force due to the displaced weight. The torque is

$$T = -WL \sin \theta - (ka \sin \theta)a$$

or, if θ is small,

$$T = -WL\theta - ka^2\theta$$

The moment of inertia is approximately

$$I = \frac{W}{g} L^2$$

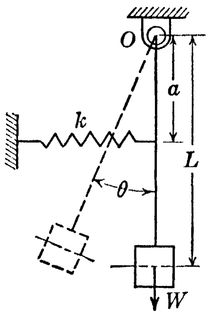


FIG. 2·8.

so that the condition for equilibrium gives

$$-WL\theta - ka^2\theta = \frac{W}{g} (L^2) \frac{d^2\theta}{dt^2}$$

or

$$\frac{d^2\theta}{dt^2} + \frac{WL + ka^2}{WL^2} g\theta = 0 \qquad [2\cdot41]$$

The frequency is then

$$f = \frac{1}{2\pi}\sqrt{\left(\frac{WL + ka^2}{WL^2}\right)g} = \frac{1}{2\pi}\sqrt{\frac{g}{L} + \frac{kg}{W}\left(\frac{a}{L}\right)^2} \qquad [2\cdot42]$$

Illustrative Problem. A pendulum such as in Fig. 2·8 has a one-pound weight at a distance of 15 in. from the support and a spring with a k value of 5 lb/in. 10 in. from the support. Determine the natural frequency of the system. What is the period of vibration?

Solution. The natural frequency of such a system can be determined by substituting in equation 2·42, giving

$$f = \frac{1}{2\pi}\sqrt{\left(\frac{1(15) + 5(10^2)}{1(15^2)}\right)386} = 29.7 \text{ cycles/sec}$$
$$= 1780 \text{ cycles/min}$$

Since the period is the time for one cycle, it is $\frac{1}{f} = \frac{1}{29.7} = 0.037$ sec.

2·14. Torsional Vibration of Spring-Mounted Machine. Figure 2·9 shows another torsional vibration system. If the machine is mounted on springs as indicated, it can vibrate about its center of rotation. When I is the moment of inertia of the mass about the center of gravity and k is the spring constant for the springs on each side, the restoring torque caused by a small angular deflection θ can be written as

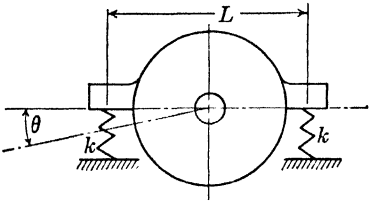


FIG. 2·9.

$$T = -k\frac{L^2}{2}\theta$$

so that the condition for equilibrium gives

$$I\frac{d^2\theta}{dt^2} + k\frac{L^2}{2}\theta = 0 \qquad [2\cdot43]$$

giving a natural frequency of

$$f = \frac{1}{2\pi}\sqrt{\frac{L^2k}{2I}} \text{ cycles/sec} \qquad [2\cdot44]$$

Illustrative Problem. The engine of an airplane is mounted as indicated in Fig. 2·9 so that it can oscillate about its center of gravity but cannot move in any other plane. If the moment of inertia is 100 in.-lb-sec^2, and two springs, each having a value of 1250 lb/in., are used on each side, spaced 20 in. apart, determine the natural frequency of this system.

Solution. The effective spring constant to be used on each side is 2(1250) = 2500 lb/in. The natural frequency as given by equation 2·44 is

$$f = \frac{1}{2\pi}\sqrt{\frac{2500(20)^2}{2(100)}} = 11.2 \text{ cycles/sec}$$

2·15. Beam and Weight. Another type of free vibrating system is the beam shown in Fig. 2·10. If the end of the beam is deflected an amount x, the energy stored in the beam will equal the average force times the deflection. The force at the end of a cantilever beam needed to deflect the end a distance x is

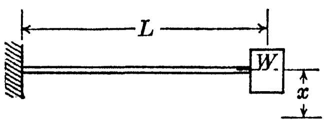


FIG. 2·10.

$$F = \frac{3EIx}{L^3}$$

The average force is one-half of this, and the potential energy stored at any deflection x is

$$PE = \frac{1}{2}\frac{3EIx}{L^3}(x) = \frac{3}{2}\frac{EIx^2}{L^3}$$

The change in potential energy due to the change in position of the weight is neglected as long as the weight exerts a constant effect on the system. This is justified because only the energy variation from the equilibrium position is of interest. This was shown to be true analytically in section 2·8 for a simple spring and weight system by changing variables. It can also be shown by referring to the spring force-deflection diagram of Fig. 2·2. The change in variables can be deduced to mean that the zero position is moved from an unstressed position to the equilibrium position. Thus it is seen that the variation depends only on the energy change relative to the equilibrium position. This simplification cannot be used where the weight has a variable effect on the system, such as a vertical pendulum.

The kinetic energy stored in the weight W is

$$KE = \frac{1}{2}\left(\frac{W}{g}\right)\left(\frac{dx}{dt}\right)^2$$

The sum of the potential and kinetic energy of the system is a constant so that we can write

$$\frac{3}{2}\frac{EIx^2}{L^3} + \frac{1}{2}\frac{W}{g}\left(\frac{dx}{dt}\right)^2 = C$$

If this expression is differentiated with respect to time and the resulting equation simplified, the following relation is obtained:

$$\frac{W}{g}\frac{d^2x}{dt^2} + \frac{3EI}{L^3}x = 0 \qquad [2\cdot45]$$

giving the frequency equal to

$$f = \frac{1}{2\pi}\sqrt{\frac{3EIg}{WL^3}} \text{ cycles/sec} \qquad [2\cdot46]$$

Illustrative Problem. Tachometers consisting of a thin steel strip having a weight at one end and rigidly fastened to a frame at the other are used for indicating speed. When the frequency of vibration of the machine or structure to which they are connected equals the natural frequency of the loaded strip, it will vibrate. One such member consists of a strip of spring steel 0.030 in. thick, $\frac{1}{4}$ in. wide, and 2 in. long. What weight should be placed on the end of this beam so that the natural frequency will be 1800 rpm, or 30 cycles/sec?

Solution. The sectional moment of inertia of the beam is

$$I = \tfrac{1}{12}(\tfrac{1}{4})(0.030)^3 = 56(10^{-8})$$

From equation 2·46

$$f = \frac{1}{2\pi}\sqrt{\frac{3EIg}{WL^3}}$$

$$W = \frac{3EIg}{L^3(2\pi f)^2} = \frac{3(30)10^6(56)10^{-8}(386)}{2^3(2\pi)^2 30^2} = 0.0685 \text{ lb}$$

2·16. General Case. The many examples which have been considered are in many respects similar. It is possible to generalize on such simple systems of one degree of freedom. The natural frequency for such systems can be written as

$$f = \frac{1}{2\pi}\sqrt{\frac{\text{Spring constant}}{\text{Mass vibrated}}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}} \text{ cycles/sec} \qquad [2 \cdot 47]$$

When it is expressed in this form it is easier to see why the U tube and pendulum are independent of the weight involved since the weight appears in both the restoring force and the mass and will therefore cancel out.

Illustrative Problems

1. A weight W is supported by two springs in parallel, as indicated in Fig. 2·11. If k_1 is the spring constant for one spring and k_2 for the other, determine the natural frequency of this system. What is the equivalent spring constant?

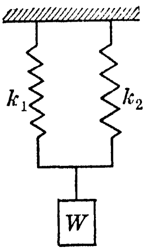


FIG. 2·11.

Solution. If the springs are in parallel, the restoring force encountered when the weight is deflected a unit distance is $k_1 + k_2$. The mass to be vibrated is W/g. From equation 2·47 we have

$$f = \frac{1}{2\pi}\sqrt{\frac{(k_1 + k_2)g}{W}}$$

The equivalent spring constant is the sum of the two, or

$$k = k_1 + k_2$$

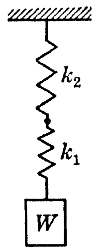


FIG. 2·12.

2. A weight is supported by two springs in series as shown in Fig. 2·12. If these springs have constants k_1 and k_2 pounds per inch, determine the natural frequency of the system. What is the equivalent spring constant?

Solution. If a force equal to W is applied to the system, the weight W will move through a distance

$$\frac{W}{k_1} + \frac{W}{k_2}$$

If k is the equivalent spring constant, the distance the weight W will move under force W will be W/k. By equating these two relations we can obtain

$$\frac{W}{k} = \frac{W}{k_1} + \frac{W}{k_2}$$

giving

$$k = \frac{k_1 k_2}{k_1 + k_2}$$

The natural frequency can be determined from equation 2·47, giving

$$f = \frac{1}{2\pi}\sqrt{\frac{k_1 k_2 g}{W(k_1 + k_2)}}$$

2·17. Effect of Distributed Mass. The discussions which have been given for springs and shafts have all neglected the effect of the weight of the spring and inertia of the shaft on the natural frequency. It has been assumed in these analyses that the weight of the spring or inertia of the shaft is small in comparison with the concentrated values. The simplified solution can be used, in most cases, with a sufficient degree of accuracy. The effect of this mass on the frequency can be calculated if desired by using energy methods.

The effect of the inertia of the shaft in the shaft and disk problem can be closely approximated using the energy method discussed in section 2·7. The potential energy stored in the shaft is given by the relation

$$PE = \tfrac{1}{2}k_t\theta^2$$

The kinetic energy will be made up of that of the disk and that of the shaft. If I is the moment of inertia of the disk and I_s is the moment of inertia per unit length of the shaft, we could write

$$KE = \frac{1}{2} I \left(\frac{d\theta}{dt}\right)^2 \text{ for the disk}$$

The angular velocity of the shaft equals that of the disk at the disk and is zero at the support. The angular velocity at a distance y from the support is $\dfrac{y}{L}\dfrac{d\theta}{dt}$. The kinetic energy of a length dy is $\frac{1}{2}I_s dy\left(\dfrac{y}{L}\dfrac{d\theta}{dt}\right)^2$.

The total kinetic energy of the shaft is given by

$$KE = \int_0^L \frac{1}{2} I_s \left(\frac{y}{L}\frac{d\theta}{dt}\right)^2 dy = \frac{1}{2}\frac{I_s}{L^2}\left(\frac{d\theta}{dt}\right)^2 \int_0^L y^2 dy$$

$$= \frac{1}{2}\frac{I_s L}{3}\left(\frac{d\theta}{dt}\right)^2$$

The total kinetic energy of the system is

$$KE = \frac{1}{2} I \left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}\left(\frac{I_s L}{3}\right)\left(\frac{d\theta}{dt}\right)^2 = \frac{1}{2}\left(I + \frac{I_s L}{3}\right)\left(\frac{d\theta}{dt}\right)^2$$

It can be seen that the inertia of the shaft can be taken into account by adding one-third of the inertia of the shaft to the disk.

It can also be shown in a similar manner that the weight of the spring in a weight and spring system can be taken into account by adding one-third of the weight of the spring to the concentrated weight being vibrated.

These results apply within the limit of accuracy normally required even though the weight of the spring or inertia of the shaft is equal to that of the weight or disk.

FORCED VIBRATIONS WITHOUT DAMPING

2·18. Spring and Weight. It is logical to continue with the study of forced vibrating systems with no damping. The combination of the natural vibration and the forced vibration is particularly important. A typical example of this is a weight suspended on a spring as shown in Fig. 2·13.

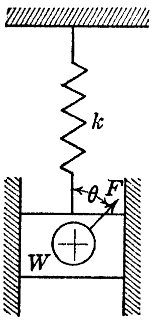


FIG. 2·13.

In order to restrict the motion to one degree of freedom it is assumed that motion is permitted in one direction only. Assume that a weight is hung on the spring and that a vertical harmonic force acts on the weight as shown. This might be a motor with an unbalanced rotor running at a constant speed. Then, if the motor is guided so that it can move only in a vertical direction, one component ($F \sin \theta$) of this unbalanced centrifugal force will be absorbed by the guides while the other component ($F \cos \theta$) will produce the force which will tend to make the motor vibrate in a vertical direction. If ν is the frequency of the unbalanced force in radians per second, the unbalanced force can be written as $F \cos \nu t$. The vertical component of the unbalanced force can be expressed in terms of either the cosine or sine, depending on whether θ is measured from the vertical or horizontal axis. The general equation of motion of this system can be written as follows:

$$\Sigma F_x = \frac{W}{g} a_x$$

$$-W - kx_1 + F \cos \nu t = \frac{W}{g} \frac{d^2 x_1}{dt^2} \qquad [2\cdot48]$$

or

$$\frac{W}{g}\frac{d^2x_1}{dt^2} + kx_1 + W = F\cos\nu t$$

or, if we let $x = x_1 - \dfrac{W}{k}$ = displacement from the equilibrium position,

$$\frac{W}{g}\frac{d^2x}{dt^2} + kx = F\cos\nu t \qquad [2\cdot 49]$$

This last equation is identical with the equation for a free vibration except that the right-hand term ($F\cos\nu t$) has been introduced. The solution for a differential equation of this type is equal to the solution when the right-hand member is zero, plus a particular integral or complementary solution. The solution when the right-hand member is zero is the same as equation 2·11. If we let Y represent the particular integral or complementary solution, we can indicate the complete solution as

$$x = A\sin\sqrt{\frac{kg}{W}}\,t + B\cos\sqrt{\frac{kg}{W}}\,t + Y \qquad [2\cdot 50]$$

When the right-hand member is of the form $F\cos\nu t$ or $F\sin\nu t$, Y, the particular integral, will be of the form (see any text on differential equations)

$$Y = C\sin\nu t + D\cos\nu t \qquad [2\cdot 51]$$

where C and D are constants to be determined by substituting this value of Y and its derivatives in equation 2·49. By doing this we find that

$$\frac{W}{g}[-C\nu^2\sin\nu t - D\nu^2\cos\nu t] + k[C\sin\nu t + D\cos\nu t] = F\cos\nu t$$

$$\left[-\frac{C\nu^2W}{g} + Ck\right]\sin\nu t + \left[Dk - \frac{D\nu^2W}{g}\right]\cos\nu t = F\cos\nu t \quad [2\cdot 52]$$

Since there is no sine term on the right-hand side of the equation, the coefficient of $\sin\nu t$ must be equal to zero, giving

$$-C\nu^2\frac{W}{g} + Ck = 0$$

Therefore

$$C = 0$$

Equating coefficients of $\cos \nu t$, we have

$$Dk - \frac{D\nu^2 W}{g} = F$$

$$D = \frac{F}{k - \frac{W}{g}\nu^2} \qquad [2\cdot53]$$

The value of Y then becomes

$$Y = \frac{F}{k - \frac{W}{g}\nu^2}\cos \nu t \qquad [2\cdot54]$$

The complete solution is given by

$$x = A \sin \omega_n t + B \cos \omega_n t + \left[\frac{F}{k - \frac{W}{g}\nu^2}\right]\cos \nu t \qquad [2\cdot55]$$

where $\omega_n = \sqrt{\frac{kg}{W}}$.

If we take a case where $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$, the constants may be evaluated as before, giving

$$A = 0$$

$$B = -\frac{F}{k - \frac{W}{g}\nu^2}$$

which, when substituted in equation 2·55, gives

$$x = -\left[\frac{F}{k - \frac{W}{g}\nu^2}\right]\cos \omega_n t + \left[\frac{F}{k - \frac{W}{g}\nu^2}\right]\cos \nu t \qquad [2\cdot56]$$

$$x = \left[\frac{F}{k - \frac{W}{g}\nu^2}\right](\cos \nu t - \cos \omega_n t) \qquad [2\cdot57]$$

The behavior of this equation can be interpreted graphically by considering what happens to the system under the action of the force F as t increases. From equation 2·57 we can see that the motion of the motor

as given by x, at any frequency of the forced vibration, is the factor $\dfrac{F}{k - \dfrac{W}{g}\nu^2}$ multiplied by a difference of the two cosine terms. This result can be obtained graphically by plotting each term as given by equation 2·56 and adding the result. Figure 2·14 shows an example where

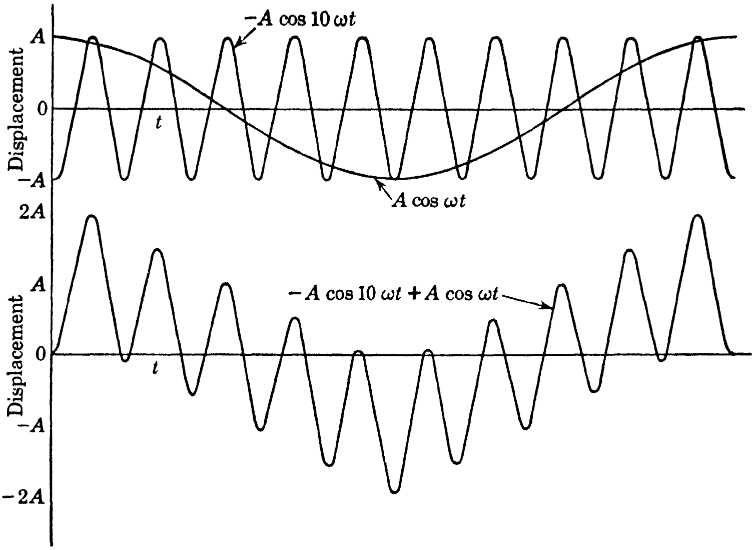


FIG. 2·14.

the impressed frequency is small compared with the natural frequency. In this figure the two terms $\left[\dfrac{F}{k - \dfrac{W}{g}\nu^2}\right]\cos\omega_n t$ and $\left[\dfrac{F}{k - \dfrac{W}{g}\nu^2}\right]\cos\nu t$ have been plotted along with the algebraic sum.

When the two frequencies are nearly alike, a condition known as beating will occur as indicated in Fig. 2·15. Since the two frequencies are slightly different they will at times tend to cancel each other and at other times to add. At the higher frequencies a phenomenon similar to that shown in Fig. 2·14 will be found except that the natural frequency will be low compared to the impressed frequency.

Illustrative Problem. A spring-mounted variable speed motor as shown in Fig. 2·13 is subjected to a dynamic force of 40 lb at 500 rpm resulting from an

unbalanced rotor. If the motor weighs 100 lb and is supported by a spring having a k value of 2000 lb/in., determine the critical speed of the system and the maximum amplitude of displacement of the vibration at 500 rpm of the motor. Assume no damping.

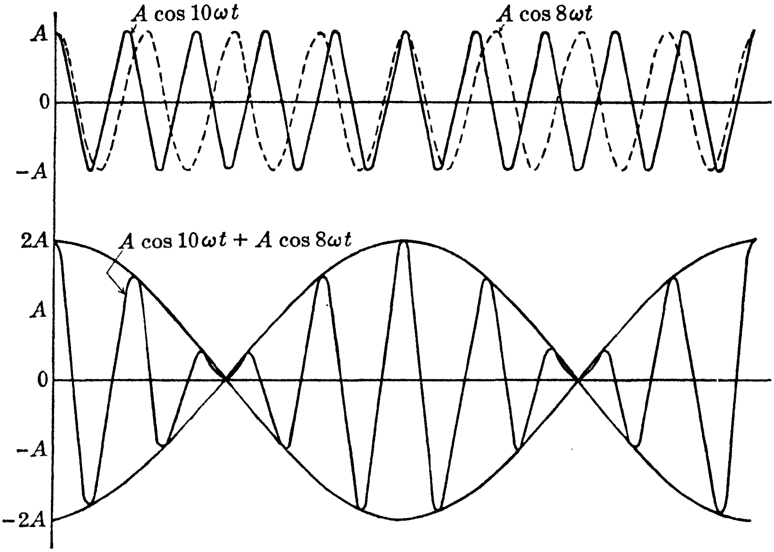


FIG. 2·15.

Solution. The critical speed occurs when the frequency of the forced vibration equals the natural frequency of the system. The natural frequency, given by equation 2·21, is

$$f = \frac{1}{2\pi}\sqrt{\frac{kg}{W}} = \frac{1}{2\pi}\sqrt{\frac{2000(386)}{100}} = 14 \text{ cycles/sec}$$

$$= 840 \text{ cycles/min}$$

When the motor speed equals 840 rpm, resonance will occur and result in excessive amplitudes.

The displacement of the vibration, given by equation 2·57, is

$$x = \left[\frac{F}{k - \dfrac{W}{g}\nu^2}\right](\cos \nu t - \cos \omega_n t)$$

where

$$\nu = \frac{2\pi N}{60} = \frac{\pi}{30}(500) = 52.5 \text{ radians/sec}$$

If these values are substituted in equation 2·57, we get for $N = 500$ rpm

$$x = \left[\frac{40}{2000 - \dfrac{(100)52.5^2}{386}}\right](\cos 52.5t - \cos 88t)$$

$$x = 0.0311(\cos 52.5t - \cos 88t)$$

To find the maximum amplitude of the vibration we could take the derivative of the above expressions for x and set it equal to zero. This would show that the amplitude is a maximum when $\nu \sin \nu t = \omega_n \sin \omega_n t$. From Fig. 2·14 or 2·15 we see that when the two motions, $\cos \nu t$ and $\cos \omega_n t$, are slightly different they will at some time get into phase for a short time, and at that time the amplitude will obviously approach the sum of the two waves. The maximum amplitude *without damping* will then be nearly 0.0622 in. at 500 rpm.

From a practical viewpoint the above conditions are impossible because there is always some damping where there is motion. If there is the slightest amount of damping the steady state conditions will not be those for undamped vibrations but will correspond to those set forth in the succeeding chapter. Because undamped forced vibrations have no practical applications except for transient conditions they will not be given further consideration.

PROBLEMS

2·1. A weight of 10 lb when hung on a helical spring causes a static deflection of $\frac{7}{8}$ in. What is the natural frequency of this system? What is the period of oscillation?

2·2. A weight of 24 lb is hung on a steel spring. The coil diameter $D = 1\frac{1}{2}$ in., the wire diameter $d = \frac{1}{8}$ in., and there are 18 active coils. What is the natural frequency of vibration if spring mass is neglected?

2·3. A mercury manometer is used to measure the air pressure in an air pipe. The bore of the tube is $\frac{1}{4}$ in., and it is filled with enough mercury to give a 22-in. column length. The specific gravity of mercury is 13.6. What is the natural frequency of the mercury in the tube?

2·4. A shaft that is fixed at one end as shown in Fig. 2·7 has a solid cylindrical disk with a moment of inertia $I = 6$ in.-lb-sec^2. It requires 114 in.-lb torque applied at the disk to produce a twist of one radian. What is the natural frequency of this system in cycles per minute?

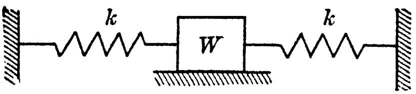


FIG. P2·5.

2·5. Each of the two springs shown in Fig. P2·5 has a spring constant $k = 2.6$ lb/in., and the weight $W = 3$ lb. What is the natural frequency of the system? If the weight is displaced 1.75 in. from its equilibrium position, what is the velocity of the weight when it goes through its equilibrium position?

2·6. Determine the modulus of elasticity for a spring for which the following information has been determined: The spring weighs 2.8 oz and, in addition, supports a weight of 6.5 oz. When it is displaced from its equilibrium position and allowed to vibrate, its natural frequency is 55 cycles/min. There are 202 active coils in the spring, the mean coil diameter is $\frac{25}{32}$ in., and the diameter of the spring wire is 0.04 in. Compare the results when the spring weight is neglected and when it is considered.

2·7. Determine the spring constant k when a load W is applied at the crosspoint of two identical beams as shown in Fig. P2·7. Deflection of a single beam is $WL^3/48EI$.

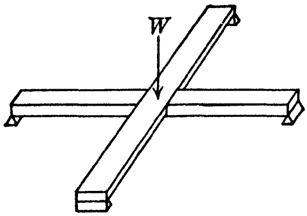


FIG. P2·7.

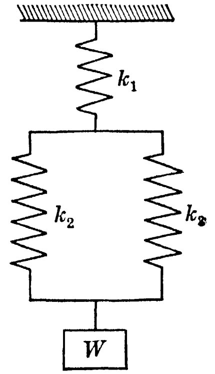


FIG. P2·8.

2·8. A weight W is suspended from a spring system as shown in Fig. P2·8. Determine the equivalent spring constant of this system. Assume linear motion for all parts of the system.

2·9. In problem 2·8 what will the natural frequency be if $k_1 = 20$, $k_2 = 28$, and $k_3 = 15$ lb/in.? The weight W is equal to 36 lb.

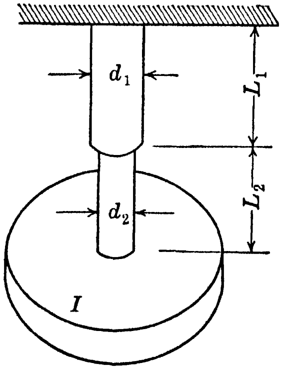


FIG. P2·10.

2·10. A disk having a moment of inertia I is mounted on the end of a stepped steel shaft as shown in Fig. P2·10. What is the equivalent spring constant of the system?

2·11. If in problem 2·10 $d_1 = 1$ in., $d_2 = \frac{1}{2}$ in., $L_1 = 24$ in., and $L_2 = 8$ in., what will be the value of the spring constant? What is the natural frequency if the disk has a moment of inertia $I = 16$ in.-lb-sec^2?

2·12. A car weighing 3800 lb deflects its springs 6.85 in. under static conditions. What is the natural frequency in the vertical direction?

2·13. A radial airplane engine is mounted on six rubber isolators arranged in a ring as shown in Fig. P2·13 so that the isolator axis is tangent to the ring. These mounts allow only torsional movement about the center of the ring. The manufacturer's catalog states that each isolator deflects 0.123 in. under a load of 690 lb along the axis of the isolator. The ring has a radius of 24 in. What is the torsional spring constant?

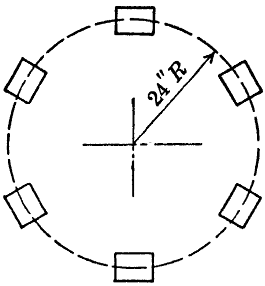


FIG. P2·13.

2·14. If the radial engine in problem 2·13 has a moment of inertia of 8000 in.-lb-sec^2, determine the natural frequency of torsional vibration.

2·15. An inverted pendulum is supported by two springs as shown in Fig. P2·15. Each spring has a spring constant equal to k. If the weight of the mass at A is W, derive an expression for the frequency of small vibrations, assuming the bar is weightless.

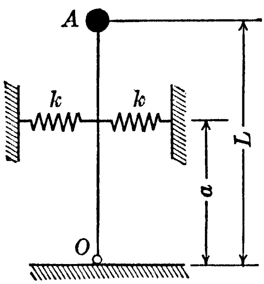


FIG. P2·15.

2·16. The pendulum of a grandfather's clock makes a complete swing in 2 sec. Determine the weight and length of pendulum if the weight on the pendulum is to be made of steel having a density of 0.280 lb/in.^{3}

2·17. A 4-in. cubical wood block having a specific gravity of 0.5 as compared with water is placed in a tank of water. It is depressed and then released. If damping is neglected, what will be the natural frequency of the block? What would its frequency be if it is placed in a tank of mercury having a specific gravity of 13.6?

2·18. The pressure indicator schematically shown in Fig. P2·18 is set up with a spring having a spring constant k = 120 lb/in. The mass of the piston, rod, etc., is about 0.1 lb. What is the natural frequency of the indicator?

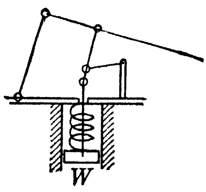


FIG. P2·18.

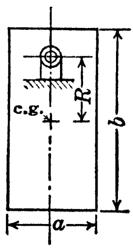


FIG. P2·19.

2·19. A large plate is pivoted near one end as in Fig. P2·19. The weight W = 75 lb has its center of gravity a distance R = 24 in. from the pivot point. The plate is 20 in. wide and 54 in. long. What is the natural frequency for small oscillations?

2·20. A spring with a spring constant $k = 1.2$ lb/in. and a weight $W = 0.8$ lb are being lowered with a uniform velocity 1.6 in./sec. Find the maximum tensile force acting on the spring if the upper end of the spring is suddenly stopped. What is the natural frequency of the system?

2·21. In Fig. P2·21 neglect the weight of the hinged rigid bar OA and the dimensions of the ball of weight W which is attached to it at A. Calculate the frequency of the system.

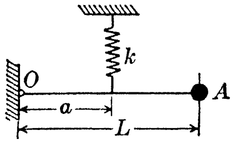


FIG. P2·21.

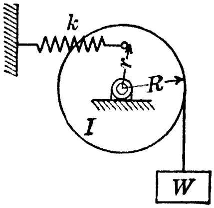


FIG. P2·22.

2·22. The weight W in Fig. P2·22 is hung by an inextensible cable on a cylinder. The motion is restrained by a spring with a modulus k. The moment of inertia of the rotor about its axis is I. If the weight is displaced a short distance downward and released, what is the natural frequency of vibration?

2·23. The steel rectangular bar of Fig. P2·23 is simply supported at the two ends and has a weight W at the center. What is the spring constant k if the weight of the bar is negligible? The cross section is $\frac{1}{2}$ in. by 1 in., the length is 42 in., and the weight $W = 16$ lb. What are the natural frequencies for vibration in the directions of its two axes? $E = 30{,}000{,}000$ lb/in.^{2}

FIG. P2·23.

2·24. In Fig. P2·24 a roller of radius r, weight W, and moment of inertia I about its center of gravity is displaced through a small angle on a cylindrical surface of radius R and then released. What is its natural frequency of oscillation?

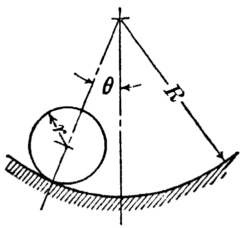


FIG. P2·24.

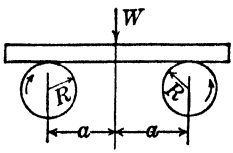


FIG. P2·25.

2·25. The coefficient of friction μ between two metals has been measured by placing a block of one material on two rollers of the other material. When the rollers rotate in opposite directions as shown in Fig. P2·25, the coefficient of friction may be determined from the frequency of the resulting vibration of the block. Derive an expression for the natural frequency of such a system. *Hint:* Displace the block a distance x from the symmetrical position and write the equations of equilibrium.

2·26. Derive an expression for the natural frequency of the system shown in Fig. P2·26. Assume that the beam between the spring and the weight is rigid.

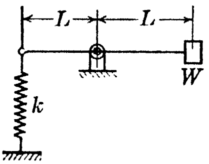


FIG. P2·26.

2·27. If the beam elasticity in problem 2·26 is to be considered, derive an expression for the natural frequency in terms of the dimensions and physical properties of the system.

2·28. A large heavy platform weighing W is hinged at one end as shown in Fig. P2·28 and is supported by a spring with a constant k at the other end. What is the natural frequency for this system?

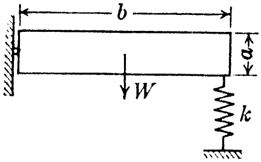


FIG. P2·28.

2·29. A connecting rod, Fig. P2·29, is supported at the wrist pin end, displaced, and allowed to oscillate. The weight is $6\frac{1}{2}$ lb, and the center of gravity is 10.75 in. from the pivot point. If the frequency of the oscillation is found to be 47 cycles/min, what is the moment of inertia about the center of gravity?

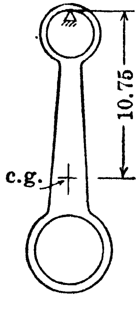


FIG. P2·29.

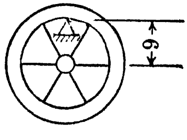


FIG. P2·30.

2·30. It is necessary to know the moment of inertia of a flywheel weighing 96 lb. Therefore, it is pivoted on the inside of the rim at a distance 9 in. from the center of the wheel as shown in Fig. P2·30. When it is displaced from the equilibrium position and released, the wheel oscillates at a rate of 32 cycles/min. What is the moment of inertia about the center of the wheel?

2·31. In Fig. P2·31 the pulley has a weight W_1 and a radius R. A weight W_2 is supported by a clevis to the pulley. The inextensible cable supporting the pulley has a spring with a spring constant k on one side. Set up an expression for the natural frequency if the moment of inertia of the pulley is included.

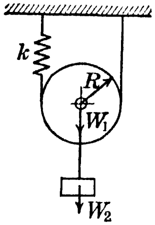


FIG. P2·31.

2·32. The natural frequency of a weight on a spring may be determined by including one-third the weight of the spring with the weight. What per cent of the weight may the weight of the spring be if the accuracy of finding the natural frequency by neglecting the weight of the spring is to be within 1%? 5%?

2·33. A steel shaft 2 in. in diameter and 6 ft long is fixed at one end and carries a steel disk 12 in. in diameter and 2 in. thick at the other end. What is the natural frequency if the effect of the shaft inertia is included?

2·34. A steel cantilever carries a weight $W = 10$ lb at its outer end. The beam is a rectangular section 2 in. wide, 1 in. deep, and 22 in. long. What is the natural frequency?

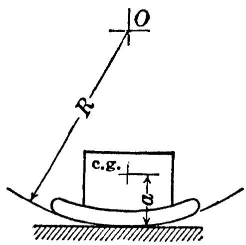


FIG. P2·35.

2·35. A cradle may be represented by Fig. P2·35. Find an expression for the natural frequency of the rocking if the center of gravity is a distance a above the floor. Assume that I is the moment of inertia about an axis through the center of gravity.

CHAPTER III

DAMPED VIBRATIONS

3·1. Introduction. Free and forced vibrations without damping represent idealized conditions. They were considered first because the solutions are simple and the methods and results are usable in the general study of damped vibrations. The selection of proper operating conditions or the design of a system to fit given operating conditions can often be made on the basis of the natural frequency. When operating conditions are near the natural frequency or resonance, damping becomes important and must be taken into account.

Damping, as considered in vibration problems, refers to those cases where some form of friction is present. In the simplest form of damping the friction force is proportional to the relative velocity between the vibrating body and some other body. This type of damping, called velocity or viscous damping, results from the motion of a body through a fluid or from the viscosity of the film of lubricant between the two bodies having relative motion. When the relative velocity is high the friction or damping becomes proportional to some higher power of the velocity. Then no simple mathematical solution can be obtained. Such cases are usually treated as though the damping were proportional to the velocity with the proportionality factor selected so as to give the same energy dissipation as the actual problem. Jacobson has demonstrated both experimentally and mathematically that such a treatment is practical.[14]

Another common form of damping is that of constant or coulomb damping. Such cases are commonly encountered in applied mechanics where the friction force is assumed to be proportional to the force between the two bodies. This proportionality factor is the coefficient of friction. Where good lubrication does not exist between two bodies, this type of damping may be attained. Although it is subject to mathematical analysis it can also be expressed as an equivalent velocity or viscous damping if desired.

The behavior of elastic bodies subjected to stress is, for most purposes, assumed to be ideal. We assume, for example, that the deflection of a bar is proportional to the load, and when the load is removed the body immediately assumes its original position. This condition holds for

normal loading where the time factor is not important. Any elastic body has some internal friction. If the bar is loaded, it requires time to assume its total deformation; and a time factor is likewise required for it to recover from its loading. If then a material is loaded over a cycle, the resulting diagram will look somewhat as shown in Fig. 3·1 rather than the ideal case normally assumed and shown in Fig. 2·2. The loop shown in Fig. 3·1 is often called the hysteresis loop. It represents the amount of internal friction or energy dissipated over a cycle of loading. If a body is vibrating, this amount of energy is dissipated each cycle. Members vibrating at a high frequency may then dissipate a considerable amount of energy and get warm or even hot if the frequency is high enough and the amplitude sufficiently large.

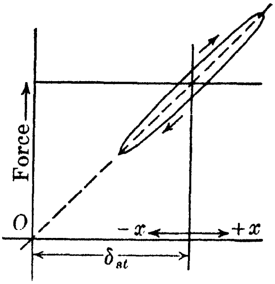


Fig. 3·1.

FREE VIBRATIONS WITH VISCOUS DAMPING

3·2. The Equation of Motion. Velocity or viscous damping is the simplest damping to consider in vibration problems. This damping, which is proportional to the velocity (dx/dt), will be called r (lb per unit velocity = lb-sec/in.). Such a problem can be represented by the spring and weight system shown in Fig. 3·2.

We can obtain the differential equation in a manner similar to that considered in the previous section by referring to the footnote in section 2·2. In this example the damping force $r(dx/dt)$ is introduced opposite the velocity. For equilibrium the equation can be written as follows, where x represents the displacement from the equilibrium position:

$$-kx - r\frac{dx}{dt} = +\frac{W}{g}\frac{d^2x}{dt^2}$$

or

$$\frac{W}{g}\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = 0 \qquad [3\cdot 1]$$

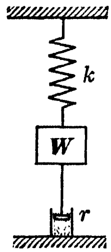


Fig. 3·2.

A value of x must now be determined that will satisfy equation 3·1.

3·3. Solution of the Differential Equation. This differential equation is seen to be very similar to equation 2·2. The solution can be obtained by following the method given there. The auxiliary equation is

$$\frac{W}{g}m^2 + rm + k = 0 \qquad [3\cdot 2]$$

The roots of this quadratic equation are

$$m_1 = -\frac{rg}{2W} + \sqrt{\left(\frac{rg}{2W}\right)^2 - \frac{kg}{W}} \qquad [3\cdot 3a]$$

$$m_2 = -\frac{rg}{2W} - \sqrt{\left(\frac{rg}{2W}\right)^2 - \frac{kg}{W}} \qquad [3\cdot 3b]$$

The best form in which to write the solution will depend upon the relative value of $(rg/2W)^2$ and kg/W. If $(rg/2W)^2$ is larger than kg/W, the solution may be written most conveniently in the form

$$x = Ae^{m_1 t} + Be^{m_2 t} \qquad [3\cdot 4]$$

If both roots are real and negative it may be proved that the motion given by equation 3·4 is no longer vibratory because it can only approach

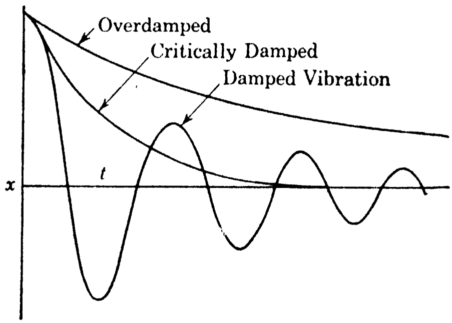


FIG. 3·3.

the equilibrium position or pass through it once. This solution, therefore, has little value.

When $(rg/2W)^2$ is equal to kg/W, the roots become real, equal, and negative. The solution given by equation 3·4 is one solution, but a more general form would be

$$x = (A + Bt)e^{mt} \qquad [3\cdot 5]$$

If for a particular case $x = x_0$ and $dx/dt = 0$ at $t = 0$, then $A = x_0$ and $B = -x_0 m$ where $m = -(rg/2W)$.

$$x = x_0\left[1 + \frac{rg}{2W}t\right]e^{-\frac{rg}{2W}t} \qquad [3\cdot 6]$$

This too is not a vibratory motion. It is of value because it is a limiting condition as will be shown in the section on critical damping.

In the third type of solution kg/W is greater than $(rg/2W)^2$. These

roots will be complex and unequal. The most usable form for the solution is

$$x = e^{-\frac{rg}{2W}t}\left[A \cos \sqrt{\frac{kg}{W} - \left(\frac{rg}{2W}\right)^2}\, t + B \sin \sqrt{\frac{kg}{W} - \left(\frac{rg}{2W}\right)^2}\, t\right] \quad [3\cdot 7]$$

This represents a vibratory motion because the term inside the brackets is a simple harmonic motion. A and B are arbitrary constants determined by the initial conditions. The three conditions are plotted in Fig. 3·3 for the conditions of $x = x_0$, $dx/dt = 0$, when $t = 0$. It will be noted that the curve for the overdamped condition never reaches the zero position. The second or critical damped condition gives a curve which settles to zero rather quickly. This fact is utilized in most instruments to eliminate excessive time in reaching the final position. The vibration in the third solution passes through zero several times.

3·4. Critical Damping. If the term under the radical sign in equations 3·3*a* and 3·3*b* is zero or positive, it has been shown that vibratory motion is impossible. Vibratory motion is possible only when this value is negative. The limiting case of vibratory motion will then be when the quantity under the radical approaches zero. The value of damping which makes the term under the radical zero is called the critical damping. Its value can be determined by writing.

$$\left(\frac{r_c g}{2W}\right)^2 = \frac{kg}{W}$$

giving

$$r_c = \sqrt{\frac{4kW}{g}} = \frac{2k}{\omega_n} = 2\omega_n \frac{W}{g} \quad [3\cdot 8]$$

This term is often used to give a measure of the relative damping in a system. It will be used later.

3·5. Discussion of the Solution for Damped Free Vibrations. The amplitude of the vibration as given by equation 3·7 is made up of two factors. The first, $e^{-\frac{rg}{2W}t}$, represents the effect of damping on the amplitude of the vibration. As time increases this quantity decreases in size. The other factor is similar to the expression for the undamped free vibration; the only difference is a slight change in frequency. Figure 3·4 shows a plot of each of the two terms together with the amplitude curve which is the product of the two. The frequency of this sytem in radians per second is given by

$$\omega_{nd} = \sqrt{\frac{kg}{W} - \left(\frac{rg}{2W}\right)^2} = \omega_n \sqrt{1 - \left(\frac{r}{r_c}\right)^2} \quad [3\cdot 9]$$

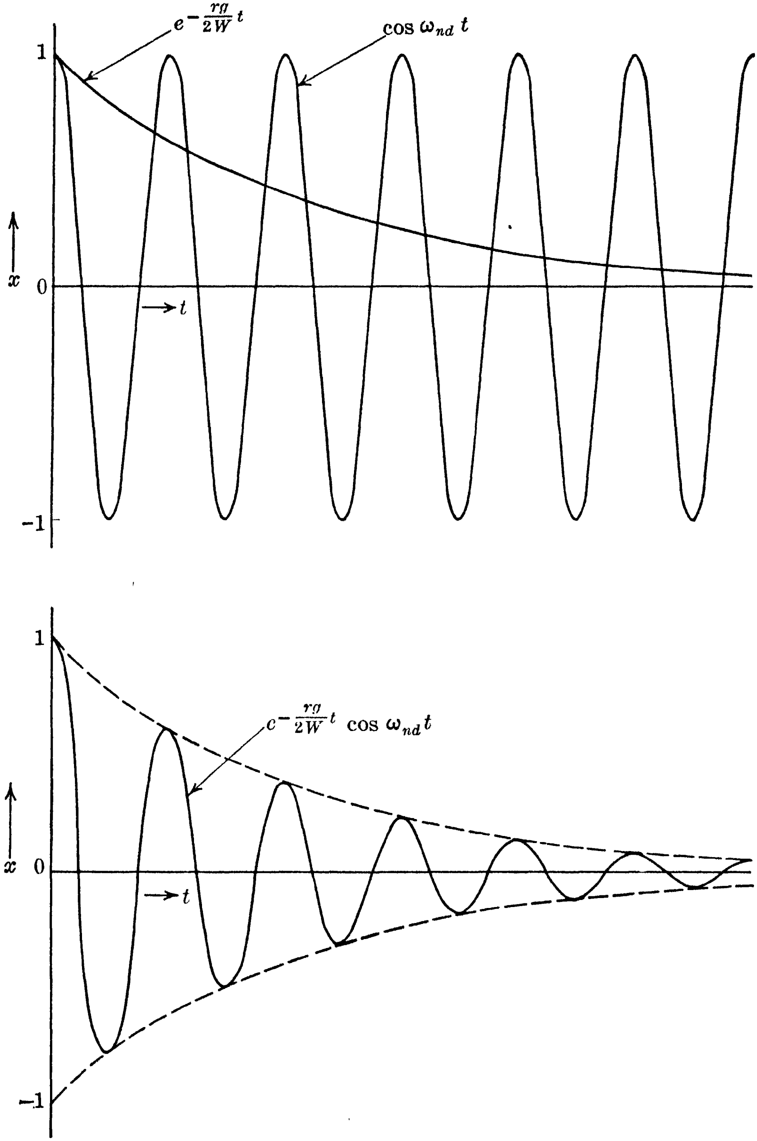

$e^{-\frac{rg}{2W}t}$
$\cos\omega_{nd}t$
1
x 0
$\rightarrow t$
-1
1
$c^{-\frac{rg}{2W}t}\cos\omega_{nd}t$
x 0
$\rightarrow t$
-1

FIG. 3·4.

where $\omega_n = \sqrt{\frac{kg}{W}}$.

The frequency of the system in cycles per second is given by

$$f = \frac{\omega_{nd}}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{kg}{W} - \left(\frac{rg}{2W}\right)^2} = \frac{\omega_n}{2\pi}\sqrt{1 - \left(\frac{r}{r_c}\right)^2} \qquad [3\cdot 10]$$

If the damping constant r is small compared with the critical damping r_c, the natural frequency of the damped vibration will be very nearly equal to the undamped frequency. With relatively high damping the frequency will be less than that of an undamped vibration. In most actual problems, the damping is smaller than $0.1r_c$ except where damping is specifically introduced.

3·6. Logarithmic Decrement. The formulas which have been developed make it possible to predict the behavior of a damped free vibration. The formulas as given do not, however, present a simple means of evaluating actual vibration records. Figure 3·5 shows a heavily damped free vibration record. This vibration follows equation 3·7 very closely. If this vibration starts from rest, that is, $t = 0$ at $x = x_0$, we will find that $A = x_0$ and $B = x_0 a/\omega_{nd}$ where $a = -(rg/2W)$ and $\omega_{nd} = \sqrt{(kg/W) - (rg/2W)^2}$. Therefore

$$x = x_0 e^{at}(\cos \omega_{nd}t - a/\omega_{nd} \sin \omega_{nd}t)$$

$$= X_0 e^{at} \cos (\omega_{nd}t - \theta) \qquad [3\cdot 11]$$

where $\theta = \tan^{-1} a/\omega_{nd}$ and $X_0 = x_0\sqrt{1 + (a/\omega_{nd})^2}$.

Two amplitudes x_1 and x_2 at times t_1 and t_2 will be

$$x_1 = X_0 e^{-\frac{rg}{2W}t_1}$$

$$x_2 = X_0 e^{-\frac{rg}{2W}t_2}$$

If the amplitudes are successive the time between them equals the period or $2\pi/\omega_{nd}$ so that $t_2 = t_1 + (2\pi/\omega_{nd})$. The second amplitude then becomes

$$x_2 = X_0 e^{-\frac{rg}{2W}\left(t_1 + \frac{2\pi}{\omega_{nd}}\right)}$$

The ratio of

$$\frac{x_1}{x_2} = e^{\frac{\pi r g}{W\omega_{nd}}} = e^{\delta} \qquad [3\cdot 12]$$

δ is called the logarithmic decrement because

$$\log_e \frac{x_1}{x_2} = \delta$$

where

$$\delta = \frac{\pi r g}{W \omega_{nd}} \qquad [3 \cdot 13]$$

where

$$\omega_{nd} = \sqrt{\frac{kg}{W} - \left(\frac{rg}{2W}\right)^2}.$$

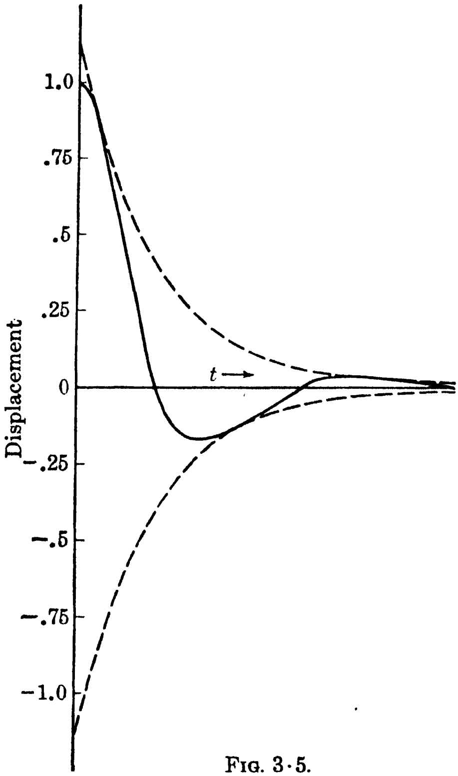


Fig. 3·5.

δ can be written in many forms by substituting the value of ω_{nd} in equation 3·13. When one is solving for δ, equation 3·13 is most convenient. When one is solving for r from an actual record, however, the following form is more useful:

$$r = 2\sqrt{\frac{kW}{g\left[\left(\frac{2\pi}{\delta}\right)^2 + 1\right]}} \qquad [3 \cdot 14]$$

δ can be determined from such a record as shown in Fig. 3·5 by taking $\log_e$ of the ratio of two successive peak values. Figure 3·6 shows other records with different values of damping.

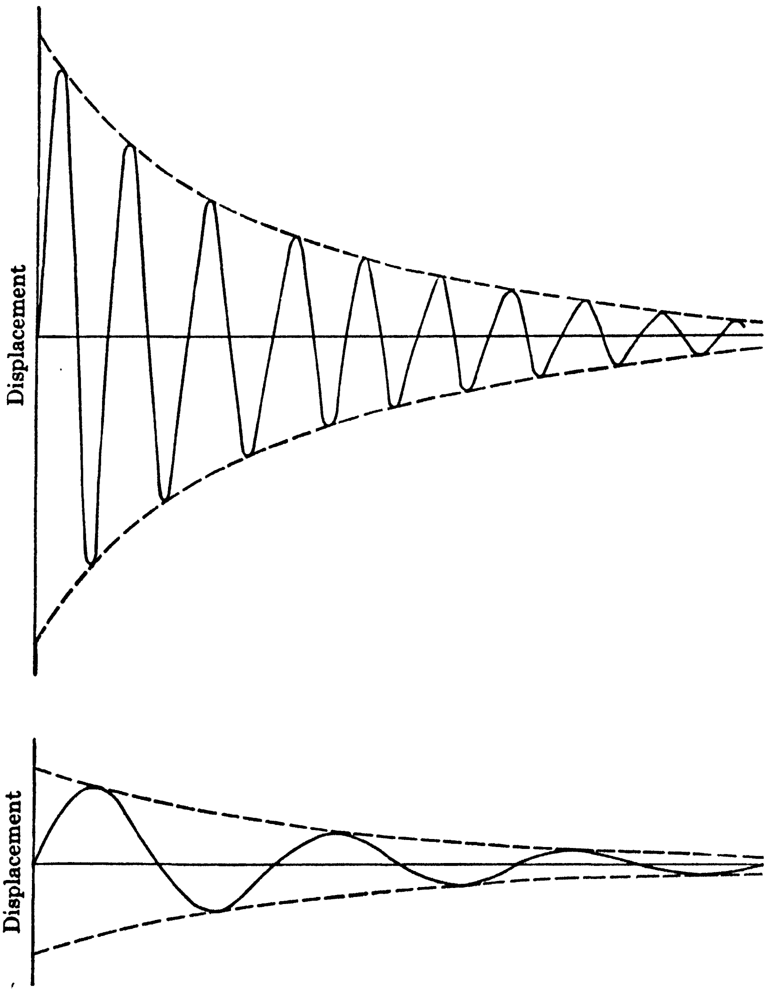


FIG. 3·6.

Illustrative Problems

1. A weight of one pound is to be supported on a spring having a stiffness of 20 lb/in. The damping constant $r = 0.01$ lb-sec/in. Determine the natural frequency of the damped and undamped vibration. Find the logarithmic decrement and the amplitude after one cycle if the initial deformation is 0.10 in.

Solution.

$$\omega_{nd} = \sqrt{\frac{kg}{W} - \left(\frac{rg}{2W}\right)^2} = \sqrt{\frac{20(386)}{1} - \left[\frac{0.01(386)}{2(1)}\right]^2}$$

$$= \sqrt{7720 - 1.93^2} = 88 \text{ radians/sec}$$

We can see from the preceding equation that damping has a negligible effect on the frequency of the vibration. The logarithmic decrement is

$$\delta = \frac{\pi(0.01)386}{1(88)} = 0.138$$

The logarithmic decrement is related to two successive peaks by the equation

$$\delta = \log \frac{x_1}{x_2} = 0.138$$

so that

$$x_1 = x_2 e^{+0.138} = 1.146 x_2$$

or

$$x_2 = 0.874 x_1$$

The amplitude of the vibration after its first cycle will be 0.874(0.10) = 0.0874 in. The amplitude at the end of the second cycle will be equal to 0.0874(0.874) = 0.0762 in.

2. A damped vibration of a weight on a spring shows the following data:

Amplitude on second cycle = 0.12 in.
Amplitude on third cycle = 0.105 in.
Spring constant k = 40 lb/in.
Weight on spring = 4 lb

Determine the damping constant.

Solution.

$$\delta = \log \frac{x_1}{x_2} = \log \frac{0.12}{0.105} = \log 1.142 = 0.131$$

From equation 3·14 we have

$$r = 2 \sqrt{\frac{40(4)}{386\left[\left(\frac{2\pi}{0.131}\right)^2 + 1\right]}} = 0.0269 \text{ lb-sec/in.}$$

3·7. Constant or Coulomb Damping. Many problems involving sliding friction can be worked on the basis of a constant coefficient of friction. For ordinary kinetics problems encountered in mechanics, this type of friction is used. Its use greatly simplifies the solutions of such problems. For vibrating systems, however, its use presents some complications. When friction is assumed proportional to the velocity,

and harmonic motion exists, the mathematical expression takes care of the sign or direction of the friction force. In the case of constant or coulomb damping, the frictional force reverses when the motion reverses, but the normal mathematical equation for coulomb damping does not take care of this situation because the force is not proportional to the displacement or any of its derivatives. It is therefore necessary to have two equations to define the motion properly. Figure 3·7 shows the forces acting when the motion is from right to left in a simple spring and weight system. In this system the differential equation of the motion is

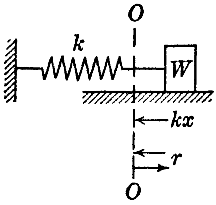


FIG. 3·7.

$$\frac{W}{g}\frac{d^2x}{dt^2} + kx - r = 0 \qquad [3\cdot 15a]$$

where $r = \mu W$ = friction force in pounds.

When the motion reverses the frictional force reverses, and the equation, when the motion is from left to right, is

$$\frac{W}{g}\frac{d^2x}{dt^2} + kx + r = 0 \qquad [3\cdot 15b]$$

If in equations 3·15a and 3·15b we substitute

$$x = x_1 + \frac{r}{k} \qquad [3\cdot 16a]$$

$$x = x_2 - \frac{r}{k} \qquad [3\cdot 16b]$$

the resulting equations are

$$\frac{W}{g}\frac{d^2x_1}{dt^2} + kx_1 = 0 \qquad [3\cdot 17a]$$

and

$$\frac{W}{g}\frac{d^2x_2}{dt^2} + kx_2 = 0 \qquad [3\cdot 17b]$$

Both these equations are the same as the equation for undamped free vibration; and the motion as described by each is seen to have the same natural frequency, which is

$$f = \frac{1}{2\pi}\sqrt{\frac{kg}{W}}\text{ cycles/sec} \qquad [3\cdot 18]$$

an identical value to that given in equation 2·21. We can therefore conclude that the natural frequency with coulomb damping is the same as the natural frequency without damping.

The effect of coulomb damping on the amplitude of the vibration is best shown by equations 3·16a and 3·16b. Physically the ratio r/k represents the extension of the spring under force r. Since successive peaks are reduced by a constant amount the envelope of the vibration

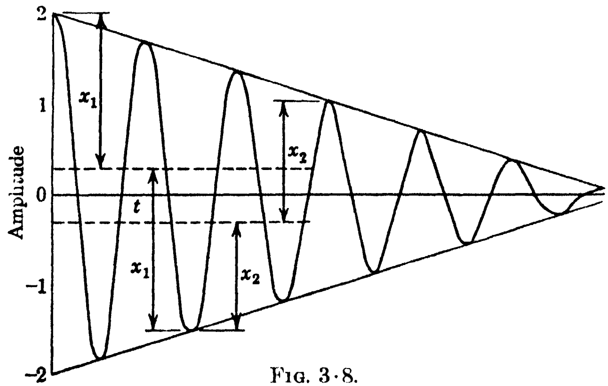


FIG. 3·8.

becomes a pair of straight lines as is indicated in Fig. 3·8. This may be expressed analytically as

$$x_1 - x_2 = 2\frac{r}{k} \qquad [3\cdot19]$$

Illustrative Problem. A weight of 2 lb is attached to a spring, as shown in Fig. 3·7, having a stiffness of 20 lb/in. The coefficient of friction between the weight and support is 0.10. Determine the frequency of vibration of the system and the amplitude at the start of the second cycle if the initial amplitude is 0.10 in.

Solution. As was indicated in equation 3·18 the frequency of this system will be identical with the frequency of an undamped system

$$f = \frac{1}{2\pi}\sqrt{\frac{20(386)}{2}} = 9.9 \text{ cycles/sec}$$

The damping force will be the product of the coefficient of friction and weight or

$$r = \mu W = 0.10(2) = 0.2 \text{ lb}$$

From equation 3·19 we see that successive points in the cycle will be decreased by an amount $\frac{2r}{k} = \frac{2(0.20)}{20} = 0.02$ in. The amplitude at the start of the second cycle will be

$$x = 0.10 - 0.04 = 0.06 \text{ in.}$$

FORCED VIBRATION WITH DAMPING

3·8. Derivation of Forced Vibration with Damping. The case of forced vibration with damping is equivalent to that discussed in section 2·18 except that the damping term is inserted. If the damping is proportional to the velocity, the amount of damping will be given by $r(dx/dt)$, and the differential equation can be derived in the same manner as equation 2·49, giving

$$\frac{W}{g}\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = F \cos \nu t \qquad [3\cdot 20]$$

where F is the magnitude of the disturbing force.

3·9. Solution of the Differential Equation. The solution of a differential equation of this type is made up of the solution when the right-hand side is zero plus the particular integral Y. We can write down the solution when the right-hand side is zero by using the results of the free vibration analysis, as given in equation 3·7. The general solution can be expressed as

$$x = e^{-\frac{rg}{2W}t}[A \cos \omega_{nd}t + B \sin \omega_{nd}t] + Y \qquad [3\cdot 21]$$

where

$$\omega_{nd} = \sqrt{\frac{kg}{W} - \left(\frac{rg}{2W}\right)^2}$$

The particular integral Y can be determined as in the case of forced vibration without damping given in section 2·18. This can be done by assuming that Y will be in the form

$$Y = C \sin \nu t + D \cos \nu t \qquad [3\cdot 22]$$

Upon differentiation we have

$$\frac{dY}{dt} = C\nu \cos \nu t - D\nu \sin \nu t$$

$$\frac{d^2Y}{dt^2} = -C\nu^2 \sin \nu t - D\nu^2 \cos \nu t$$

Since equation 3·22 is a particular solution of equation 3·20, Y and its derivatives may be substituted for x and its corresponding derivatives in equation 3·20 so that the following expression is obtained:

$$\frac{W}{g}[-C\nu^2 \sin \nu t - D\nu^2 \cos \nu t] + r[C\nu \cos \nu t - D\nu \sin \nu t] + k[C \sin \nu t + D \cos \nu t] = F \cos \nu t \qquad [3\cdot 23]$$

Since the coefficients of sine and cosine must be the same on each side of the equation, we can write

$$\left(k - \frac{W\nu^2}{g}\right)C - r\nu D = 0$$

$$r\nu C + \left(k - \frac{W\nu^2}{g}\right)D = F$$

giving

$$D = \frac{F\left(k - \frac{W\nu^2}{g}\right)}{\left(k - \frac{W\nu^2}{g}\right)^2 + (r\nu)^2} \qquad [3\cdot 24a]$$

$$C = \frac{Fr\nu}{\left(k - \frac{W\nu^2}{g}\right)^2 + (r\nu)^2} \qquad [3\cdot 24b]$$

Y then becomes equal to

$$Y = \frac{Fr\nu}{\left(k - \frac{W\nu^2}{g}\right)^2 + (r\nu)^2} \sin \nu t + \frac{F\left(k - \frac{W\nu^2}{g}\right)}{\left(k - \frac{W\nu^2}{g}\right)^2 + (r\nu)^2} \cos \nu t \qquad [3\cdot 25]$$

A sine term and a cosine term with the same frequency may be represented by two vectors at right angles, with the cosine term 90° in the

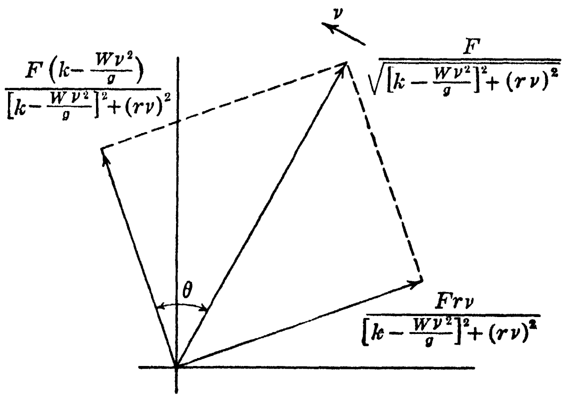


FIG. 3·9.

counterclockwise direction from the sine term as shown in Fig. 3·9. The sum of the two terms is the resultant vector so that Y can be written as

$$Y = \frac{F}{\sqrt{\left(k - \frac{W\nu^2}{g}\right)^2 + (r\nu)^2}} \cos(\nu t - \theta) \qquad [3 \cdot 26]$$

or, by rearranging,

$$Y = \frac{F}{k}\left[\frac{1}{\sqrt{\left[1 - \left(\frac{\nu}{\omega_n}\right)^2\right]^2 + \left(2\frac{r}{r_c}\frac{\nu}{\omega_n}\right)^2}}\right] \cos(\nu t - \theta) \qquad [3 \cdot 27]$$

where θ from Fig. 3·9 may be seen to be

$$\theta = \tan^{-1} \frac{r\nu}{k - \frac{W\nu^2}{g}} \qquad [3 \cdot 28]$$

The complete solution of equation 3·20 will be

$$x = e^{-\frac{rg}{2W}t}[A \cos \omega_{nd}t + B \sin \omega_{nd}t] + \frac{F \cos(\nu t - \theta)}{\sqrt{\left(k - \frac{W\nu^2}{g}\right)^2 + (r\nu)^2}} \qquad [3 \cdot 29]$$

The general solution given by equation 3·29 consists of two parts, one the forced vibration term and the other the free vibration term. As time passes, the free vibration term will disappear because the coefficient $e^{-\frac{rg}{2W}t}$ approaches zero. The steady state condition is then specified by

$$x = \frac{F \cos(\nu t - \theta)}{\sqrt{\left(k - \frac{W\nu^2}{g}\right)^2 + r^2\nu^2}} \qquad [3 \cdot 30]$$

or

$$x = \frac{F}{k} \frac{\cos(\nu t - \theta)}{\sqrt{\left[1 - \left(\frac{\nu}{\omega_n}\right)^2\right]^2 + \left(2\frac{r}{r_c}\frac{\nu}{\omega_n}\right)^2}} \qquad [3 \cdot 31]$$

From this we see that the maximum amplitude will be

$$x_0 = \frac{F}{k}\left[\frac{1}{\sqrt{\left[1 - \left(\frac{\nu}{\omega_n}\right)^2\right]^2 + \left(2\frac{r}{r_c}\frac{\nu}{\omega_n}\right)^2}}\right] \qquad [3 \cdot 32]$$

The term F/k is the deflection caused by a force equal to the disturbing force F. The other factor within the brackets is therefore a factor by

which this equivalent "static deflection" is multiplied to obtain the dynamic amplitude. For this reason the term

$$\frac{1}{\sqrt{\left[1 - \left(\frac{\nu}{\omega_n}\right)^2\right]^2 + \left(2\frac{r}{r_c}\frac{\nu}{\omega_n}\right)^2}}$$

is usually called the magnification factor.

The magnification factor has been plotted in Fig. 3·10 to show the effects of various frequency ratios on the amplitude at different ratios of damping. Three very distinct regions should be noted. At very low ratios of forced frequency to natural frequency the motion is very nearly 1, that is, very near F/k. As the frequency ratio increases the amplitude increases until the frequency ratio reaches 1. The less damping there is, the higher the amplitude. With no damping then it is seen to approach infinity when the ratio of frequencies is 1. From this we conclude immediately that damping will reduce the amplitude near the resonant frequency. Above this frequency ratio of 1 the amplitude and the magnification factor decrease. At still higher speeds the magnification factor continues to decrease and approaches zero. The phase angle between the force and the displacement, as given by equation 3·28, is shown in Fig. 3·11. With no damping the motion is in phase with the force below the critical speed and 180° out of phase above it.

Illustrative Problem. A large motor running at 1760 rpm has become unbalanced to such an extent that the unbalance force of 150 lb causes its support to vibrate. The support has a spring constant of about 30 tons/in. and a damping device that gives it a damping constant of about 200 lb-sec/in. The weight of the motor and moving part of the support is 640 lb. What would the amplitude be if there were no damping? What is the natural frequency of the support?

Solution. The amplitude of the forced vibration may be determined from equation 3·30, which is

$$x_0 = \frac{F}{\sqrt{\left(k - \frac{W\nu^2}{g}\right)^2 + (r\nu)^2}}$$

The value of ν is $2\pi 1760/60 = 184$ radians/sec. By direct substitution the amplitude is

$$x_0 = \frac{150}{\sqrt{\left(30(2000) - \frac{640(184)^2}{386}\right)^2 + [200(184)]^2}}$$

$$= \frac{150}{\sqrt{3800^2 + 36{,}800^2}} = 0.00407 \text{ in.}$$

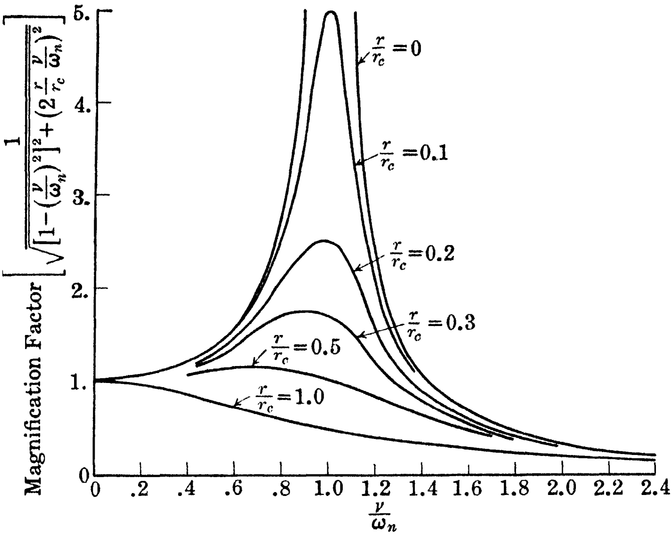


FIG. 3·10.

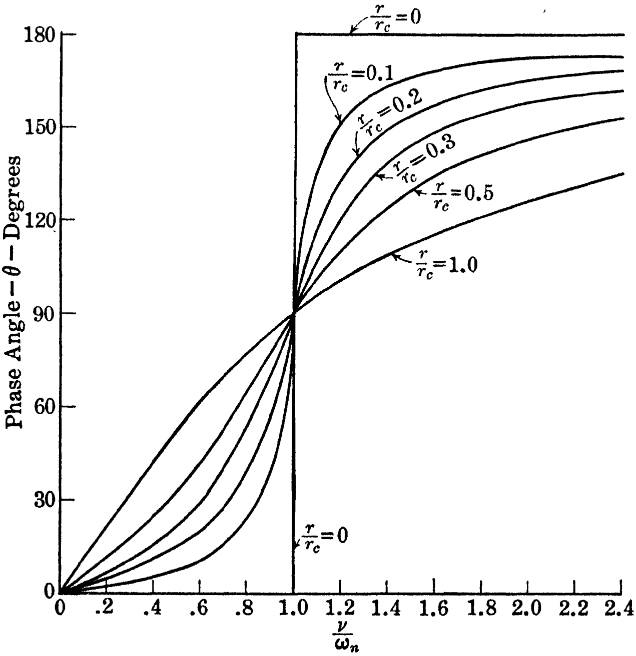


FIG. 3·11.

Without damping the last term under the radical disappears, and the amplitude becomes

$$x_0 = \frac{150}{3800} = 0.0395 \text{ in.}$$

This indicates that damping here is very valuable because the natural frequency is apparently near the operating speed. The natural frequency is

$$f = \frac{60}{2\pi}\sqrt{\frac{60{,}000(386)}{640}} = 1820 \text{ cycles/min}$$

The total force that will be transmitted to the floor is kx_0. If this force is large enough to cause vibration or discomfort somewhere, it would be necessary to reduce it by some method to be discussed in Chapter V.

STEADY STATE FORCED VIBRATIONS WITH SMALL DAMPING

3·10. Force Applied to Weight. In section 2·18 forced vibrations with no damping were considered. This case was dismissed because it retained both a free vibration and a forced vibration, a condition contrary to actual experience. In section 3·9 a solution has been obtained for forced vibration with damping. Equation 3·29 is the entire solution; but as long as there is any damping, regardless of how small it may be, the free vibration portion will disappear and we shall have a steady state condition determined by equations 3·30 or 3·31. If the damping is very small, these equations may be further reduced with negligible error to

$$x = \frac{F}{k}\left[\frac{1}{1 - \left(\frac{\nu}{\omega_n}\right)^2}\right]\cos \nu t \qquad [3\cdot 33]$$

or the amplitude is given by

$$x_0 = \frac{F}{k}\left[\frac{1}{1 - \left(\frac{\nu}{\omega_n}\right)^2}\right] = \frac{F}{k - \frac{W\nu^2}{g}} \qquad [3\cdot 34]$$

The term in brackets is also called the magnification factor. It is plotted separately in Fig. 3·12. The same three regions are apparent as where friction was considered. At a low frequency ratio of ν/ω_n the forced frequency is so slow that the kinetic energy in the mass can be absorbed in the spring with very little motion. Therefore, the deflection is very nearly the "equivalent static deflection" F/k. As the speed increases the kinetic energy increases until a resonant condition is

reached under which conditions the amplitude becomes very large. Then at very high frequencies the inertia effect builds up as the square of the frequency and soon reduces the effectiveness of the impressed force. This results in the weight being unable to follow rapid application of force. There will then be an appreciable reduction in amplitude.

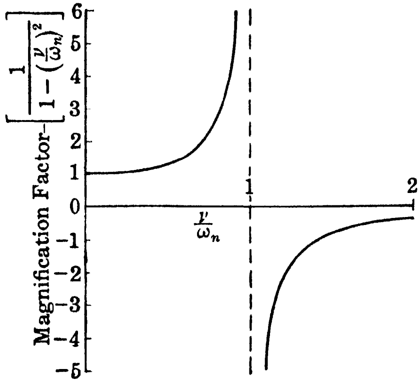


FIG. 3·12.

This may be illustrated very simply by a rope swing. If a force is applied very slowly the swing is deflected an amount approximately F/k. If the frequency of applying the force is increased the amplitude of motion is greatly increased. Now if the same force were applied with an air hammer at a high frequency the swing would hardly move.

Illustrative Problem. A motor and pump operating at 1200 rpm are mounted upon a sub-base. The whole assembly weighs 200 lb and is mounted upon springs having a total spring constant equal to 500 lb/in. If only vertical motion is permitted, what would the amplitude of motion be if an unbalance force of 2.56 lb were present? Is it operating in the region of least amplitude?

Solution. The maximum by equation 3·34 equals

$$x_0 = \frac{F}{k - \dfrac{W\nu^2}{g}} = \frac{2.56}{500 - \dfrac{200}{386}\left(\dfrac{2\pi 1200}{60}\right)^2} = -0.00033 \text{ in.}$$

The minus sign indicates that the combination is operating above the natural frequency because the displacement is opposite the force as is indicated in Fig. 3·12. The actual amount of motion will be twice the amplitude. The motor is operating in the region of least amplitude.

3·11. Spring and Weight with Motion of Support. In some types of problems, particularly vibration measuring instruments, a motion or force is transmitted to the support of an elastic system which will tend to set up vibrations in the system. Such a system may be represented as shown in Fig. 3·13, where the spring support is given simple harmonic motion represented by

$$x_1 = x_0 \cos \nu t \qquad [3\cdot35]$$

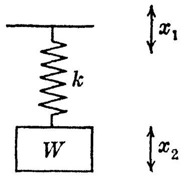


FIG. 3·13.

The spring force acting on the weight W will be $k(x_2 - x_1)$ where x_2 is the displacement of the weight W from its static position when the crank is horizontal and x_1 is at its center position. Applying the conditions for equilibrium and simplifying, we find the equation to be

$$\frac{W}{g}\frac{d^2x_2}{dt^2} + k(x_2 - x_1) = 0$$

or

$$\frac{W}{g}\frac{d^2x_2}{dt^2} + kx_2 = kx_1$$

or, by substituting equation 3·35,

$$\frac{W}{g}\frac{d^2x_2}{dt^2} + kx_2 = kx_0 \cos \nu t \qquad [3\cdot36]$$

If we let $kx_{01} = F$, equation 3·36 reduces to

$$\frac{W}{g}\frac{d^2x_2}{dt^2} + kx_2 = F \cos \nu t \qquad [3\cdot37]$$

Equation 3·37 is, therefore, the equation used for determining the absolute motion of the vibrating body with the force applied to the support. This motion will be the same as where the force is applied to the weight, providing the forces are the same.

If kx_{01} is substituted for F in equation 3·33 and reduced, we have as the solution for the absolute motion of the weight

$$x_2 = x_{01}\left[\frac{1}{1 - \left(\frac{\nu}{\omega_n}\right)^2}\right] \cos \nu t \qquad [3\cdot38]$$

Illustrative Problem. Find the amplitude of a weight in a measuring instrument represented by Fig. 3·13. The weight W is 2 lb, and the spring has a spring constant equal to 10 lb/in. The amplitude of displacement of the sup-

port is $\frac{1}{8}$ in., and the frequency is 1150 cycles/min. What is the relative motion between the weight and the support?

Solution. The natural frequency is determined as usual:

$$f = \frac{60}{2\pi}\sqrt{\frac{kg}{W}} = \frac{60}{2\pi}\sqrt{\frac{10(386)}{2}} = 420 \text{ cycles/min}$$

The amplitude of the weight W_2 is now determined from the solution of the basic equation as given by equation 3·38; that is,

$$x_{02} = x_{01}\left[\frac{1}{1 - \left(\frac{\nu}{\omega_n}\right)^2}\right] = \frac{1}{8}\left[\frac{1}{1 - \left(\frac{1150}{420}\right)^2}\right] = -0.0193 \text{ in.}$$

The relative motion between the support and the weight is

$$(x_{02} - x_{01}) = (-0.0193 - 0.125) = -0.144 \text{ in.}$$

These may be added arithmetically because there is no damping and the two terms are therefore in phase or 180° out of phase.

For the motion x_{02}, the negative sign indicates the motion to be opposite the displacement x_{01} of the support, or, in other words, the force leads the motion of the weight by 180°. The amplitude of the relative motion would then be the sum of the absolute values of the two relative motions, or 0.144 in. as indicated.

3·12. Force on Weight Varying with the Frequency. When a particular instrument or machine is to be operated over a wide range of speeds, it is desirable to include the change in the force with the speed. The general solution, given by equation 3·33, can be used if

$$F = \frac{W_r}{g} R\nu^2$$

where W_r = weight of unbalanced rotating mass in pounds
R = eccentricity of unbalance weight in inches
ν = rotating frequency in radians per second

The result will be

$$x_0 = \frac{W_r R\nu^2}{gk}\left[\frac{1}{1 - \left(\frac{\nu}{\omega_n}\right)^2}\right] = R\frac{W_r}{W}\left[\frac{\left(\frac{\nu}{\omega_n}\right)^2}{1 - \left(\frac{\nu}{\omega_n}\right)^2}\right] \qquad [3\cdot39]$$

where W = total vibrating weight in pounds. This equation may be represented graphically by Fig. 3·14. The three important regions are still maintained. At low speeds, where the ratio ν/ω_n is much less than one, the displacement of the weight is also less than the radius of the

unbalanced mass. Near the natural frequency the amplitude will again approach infinity, but above this speed the amplitude will decrease and approach R. For unbalanced rotors this may be interpreted as follows:

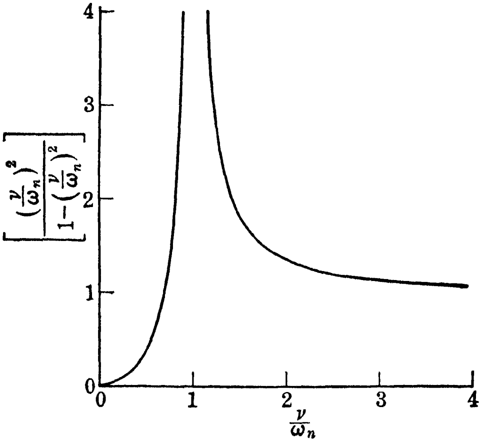


FIG. 3·14.

At the low ratios, the rotor tends to rotate near the geometric center and in phase with the force; but above the resonant speed it tends to rotate about the center of the mass, which is a distance R from the geometric center. The force now leads the displacement by 180°.

3·13. Relative Motion of a Weight and Support. When the support moves over a range of speeds and impresses a force on the system, we can look back to equation 3·38 to get the absolute amplitude of the mass. This value is of little interest to us. For measuring vibrations we are more interested in the relative motion between the support and the weight as this may be recorded easily. This relative motion is given by the equation

$$x_2 - x_1 = x_0\left[\frac{1}{1-\left(\frac{\nu}{\omega_n}\right)^2}\right]\cos \nu t - x_0 \cos \nu t \qquad [3\cdot 40]$$

The amplitude will therefore be

$$x_2 - x_1 = x_0\left[\frac{1}{1-\left(\frac{\nu}{\omega_n}\right)^2} - 1\right] = x_0\left[\frac{\left(\frac{\nu}{\omega_n}\right)^2}{1-\left(\frac{\nu}{\omega_n}\right)^2}\right] \qquad [3\cdot 41]$$

A pencil or stylus fastened to the weight may be made to mark on a paper fastened to the frame as shown in Fig. 3·15. Thus for measuring the amplitude of displacement it is desirable for the mass to stand still while the frame moves with the vibrating member. In section 3·11 it was found that the motion of the mass was very small when ν/ω_n was very large. Instruments based on this principle are known as mechanical vibrometers.

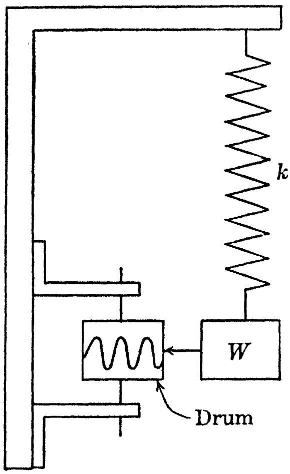


FIG. 3·15.

It is also possible to adapt a system such as is shown in Fig. 3·15 to measure accelerations. Equation 3·41 can be written as

$$x_2 - x_1 = \frac{x_0\nu^2}{\omega_n^2}\left[\frac{1}{1-\left(\frac{\nu}{\omega_n}\right)^2}\right]$$

The expression $x_0\nu^2$ is in reality the acceleration of the vibrating body to which the accelerometer is attached. Therefore,

$$A = x_0\nu^2 = (x_2 - x_1)\omega_n^2\left[1-\left(\frac{\nu}{\omega_n}\right)^2\right] \qquad [3\cdot 42]$$

Since ω_n^2 is a constant for any particular instrument, the relative displacement will be very nearly proportional to the acceleration provided the value of ν/ω_n is kept very small. This is possible by having a relatively stiff spring.

Illustrative Problem. Solve for the relative motion in the example in section 3·11.

Solution. By substituting in equation 3·44 the same answer is obtained as before.

$$x_{02} - x_{01} = \frac{1}{8}\left[\frac{\left(\frac{1150}{420}\right)^2}{1-\left(\frac{1150}{420}\right)^2}\right] = -0.144 \text{ in.}$$

3·14. Shaft and Disk. The solution of the case where a variable torque is applied to a disk and shaft system, as shown in Fig. 2·7, will be the same as equation 3·20, except that a term ($T \cos \nu t$) is added as the right-hand side of the equation. The general equation of motion for such a system will then be

$$I\frac{d^2\theta}{dt^2} + r\frac{d\theta}{dt} + k_t\theta = T\cos\nu t \qquad [3\cdot 43]$$

This equation is seen to be identical in form with equation 3·20. The solution can be obtained by substituting the appropriate equivalent symbols into equation 3·29.

PROBLEMS

3·1. In Fig. 3·2 the weight $W = 75$ lb and is connected to a spring with a modulus $k = 25$ lb/in. Attached to the weight is a piston moving in a dash pot filled with a viscous fluid. The damping force is directly proportional to the velocity of the weight and is equal to 40 lb when the velocity is 3 ft/sec. Find:

(*a*) The damping constant.
(*b*) The natural frequency of damped vibration.
(*c*) The natural frequency of undamped vibration.
(*d*) The logarithmic decrement.
(*e*) The critical damping value.
(*f*) The amplitude after one cycle if the initial amplitude is 2 in.
(*g*) The force that is exerted by the critical damping if the velocity is 3 ft/sec.

3·2. What will be the results in problem 3·1 if the damping is changed to 20 lb when the velocity is 3 ft/sec?

3·3. A body weighing 15 lb is suspended from a spring with a constant $k = 5$ lb/in. A dash pot is attached to the weight. It produces a resistance of 0.03 lb at a velocity of 1 in./sec. What is the frequency of the system? What is the ratio of two consecutive maxima? If the mass is displaced 2 in. and released, what is the amplitude 10 cycles later?

3·4. A weight of 200 lb is mounted on springs so that the static deflection is $1\frac{1}{2}$ in. As it is desirable to know some of the damping characteristics of a shock absorber it is connected to the mass and to the fixed support. The mass is then deflected 1 in. and released. At the beginning of the second cycle the amplitude is 0.73 in. If the damping is proportional to the velocity, what is the value of r/r_c and r? Do not make any approximations.

3·5. In Fig. 3·7 a weight of 42 lb slides on the supporting surface. A coefficient of friction equal to 0.18 exists between the weight and the surface. The spring has a spring constant equal to 50 lb/in. What is the natural frequency? What is the amplitude at the beginning of the second cycle if the weight were originally displaced 1 in.? What will it be at the beginning of the fourth cycle?

3·6. A shock absorber with a damping ratio $r/r_c = 0.06$ is used to dampen the vibrations of a mass weighing 340 lb that is supported on springs having a total spring constant $k = 870$ lb/in. What is the ratio between the first and second amplitudes?

3·7. A torsional system such as is shown in Fig. P3·7 has a moment of inertia $I = 8$ lb-in.-sec². The torsional spring constant of the shaft $k_t = 460$ lb-in./radian. If the torsional damping is $r_t = 25$ lb-in.-sec/radian, what will the natural frequency be for this system? If the mass were displaced 12° and then released, what would the amplitude be at the beginning of the second cycle?

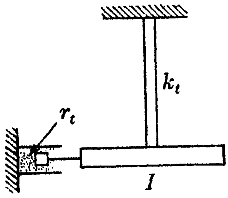


FIG. P3·7.

3·8. A weight W of 235 lb is supported on springs with a total spring constant $k = 960$ lb/in. A damping constant of $r = 6$ lb-sec/in. is inserted between the weight and the

support. If a force of 24 lb is impressed at a frequency of 100 cycles/min, what will be the amplitude of motion after it has reached a steady state? What will the amplitude be for 24 lb at 400 cycles/min? For 1200 cycles/min?

3·9. What would the results in problem 3·8 be if r were changed to 2 lb-sec/in.?

3·10. Set up the equation of forced motion for a torsional system with damping as shown in Fig. P3·7. What are the units of the terms involved? What will be the steady state solution for the amplitude?

3·11. A weight of 76 lb hanging on a spring with a spring constant $k = 115$ lb/in. has negligible damping. The weight includes a motor with an unbalance of 16 lb at a radius of 0.005 in. The motor runs at 1760 rpm. What will be the amplitude of displacement for the weight?

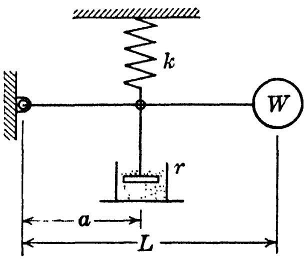


Fig. P3·13.

3·12. A trailer weighing 1600 lb deflects its springs 8.75 in. It is driven over the surface of a road which may be assumed to be a sine wave. The amplitude of the road surface is 1.5 in. or the total movement from the bottom of the wave to the crest is 3 in. If the trailer is pulled at the rate of 35 mph and the distance between succeeding crests is 40 ft, what will be the amplitude of vertical displacement for the trailer? Assume linear motion throughout.

3·13. Set up the differential equation for the motion of the weight W in Fig. P3·13.

3·14. Using the equation derived in problem 3·13 and the solution to the equation for the linear system, write the expression for the amplitude of displacement for the mass.

3·15. A disk with a moment of inertia $I = 5$ lb-in.-sec^2 is fastened to the end of a shaft with a torsional spring constant $k_t = 56$ lb-in./radian. A torsional damping of $r_t = 2$ lb-in.-sec/radian acts on the disk. Determine the following:

(*a*) The natural frequency of damped vibration.
(*b*) The natural frequency of undamped vibration.
(*c*) The logarithmic decrement.
(*d*) The critical damping.
(*e*) The amplitude of displacement at the beginning of the second cycle if the initial displacement is 10°.

3·16. A periodic torque with a maximum value $T_0 = 1500$ lb-in. is applied to a disk with a moment of inertia $I = 3.6$ lb-in.-sec^2. The disk is connected by a shaft to a large flywheel with a moment of inertia that may be considered infinite; so the system is reduced to a simple shaft and disk problem. The steel shaft connecting the two masses is $\frac{1}{2}$ in. in diameter and 52 in. long. What is the amplitude of twist in the shaft if the impressed frequency is 1390 cycles/min? What is the stress induced by the vibration?

3·17. In problem 2·13 the rubber has a damping ratio of $r/r_c = 0.06$. What will be the angular amplitude if an harmonic force with a maximum value of 30,000 lb-in. is applied at a speed of 2000 cycles/min?

3·18. A vibrometer shown in Fig. P3·18 has a period of free vibration of 2 sec. It is attached to a machine with a vertical harmonic frequency of

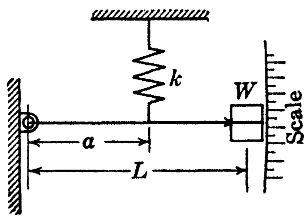


Fig. P3·18.

one cycle per second. If the vibrometer mass has an amplitude of $\frac{1}{4}$ in. relative to the vibrometer frame, what is the amplitude of vibration of the machine?

3·19. An instrument for measuring accelerations records 30 oscillations/sec. The natural frequency of the instrument is 800/sec. What is the acceleration of the machine part to which the instrument is attached if the amplitude recorded is 0.002 in.? What is the amplitude of the vibration of the machine parts?

3·20. A spring-mounted motor operates at 1760 rpm, and the unbalance force sets up a centrifugal force equal to one-eighth the weight of the motor. The static deflection $W/k = 0.005$ in. What is the ratio of the amplitude of forced vibration to the static deflection?

3·21. In the Scotch-Yoke shown in Fig. P3·21 the crank is turning at a speed of 500 rpm. The crank length is $\frac{3}{4}$ in. The static deflection of the mass on the lower end of a spring attached to Scotch-Yoke is 2 in. What is the amplitude of the forced vibration of the suspended weight?

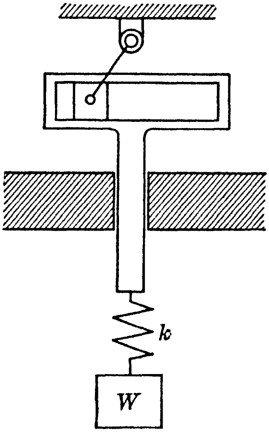


FIG. P3·21.

3·22. When an accelerometer is attached to a reciprocating crosshead the amplitude of vibration recorded is $\frac{1}{4}$ in. Find the maximum acceleration of the crosshead if the spring constant for the instrument is $k = 10$ lb/in. and the suspended weight $W = 0.20$ lb. The speed of the engine is 160 rpm.

3·23. A horizontal shaft rotates in bearings at its ends. A disk weighing 225 lb is keyed to the shaft midway between the bearings, but the center of mass of the disk is located 0.01 in. from the axis of the shaft. A static force of 1600 lb deflects the shaft 0.15 in. Neglect the weight of the shaft. Calculate the resonant speed of rotation of the shaft. If the speed of rotation is half the resonant speed, calculate the amplitude of steady state forced vibration.

3·24. A machine weighing 1250 lb and running 1760 rpm is supported on four steel springs. The spring dimensions are $d = \frac{1}{2}$ in., $D = 4$ in., and the number of coils $n = 10$. $G = 12{,}000{,}000$. The rotating mass weighs 200 lb, but the center of gravity is 0.005 in. from the center of rotation. What is the amplitude of vibration for the machine? What is the force transmitted to the foundation?

3·25. The simple disk and shaft system, such as is shown in Fig. P3·7, has a periodic torque $T = T_0 \cos \nu t$ applied to the disk. Derive the equation for the angular displacement of the disk from its equilibrium position, assuming that the steady state conditions have been reached.

CHAPTER IV

VIBRATION OF SYSTEMS WITH SEVERAL DEGREES OF FREEDOM

4·1. Introduction. The previous chapters have dealt only with the simpler types of vibration systems where the mass of the system was considered as concentrated at one point and the spring or elastic member was between the concentrated mass and support. The large number of problems which can be solved by such a simplified system justifies the attention given to it. There is, however, the broader field where there are many masses connected by elastic members. In such systems each mass can move independent of the other masses, and the system is said to be one of several degrees of freedom. Inasmuch as the solution becomes increasingly difficult as the number of masses or degrees of freedom is increased, different methods of approach are required in the solution of practical problems of this type. These methods often involve approximations. The following discussion covers the analytical solution of common cases of several degrees of freedom. Illustrations are given for tabulation and graphical methods of obtaining solutions to problems of this type. A later chapter will deal with the mobility method, which offers another means of solving problems of several degrees of freedom.

4·2. Disk and Shaft Problem with Two Degrees of Freedom. A simple type of problem which has two degrees of freedom is that represented by Fig. 4·1, where two disks having a moment of inertia I_1 and I_2 are connected by an elastic shaft having a spring constant k_t. If this shaft is supported in bearings having no friction and the disks are twisted with respect to each other, the system will vibrate when the disks are released. Our immediate problem is to find the frequency of vibration.

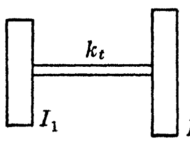


FIG. 4·1.

The system shown in Fig. 4·1 differs from that shown in Fig. 2·7 in that the disk I_2 in Fig. 2·7 was of infinite size. In that example we assumed that the support had no motion. In the present example both disks will move. Considered from an energy standpoint, the vibration will consist of a transfer of energy from the shaft to the disks and vice versa. Since energy can be stored in the shaft only as a result of relative motion of the system, it follows that the motion of the two disks

must take place with the same frequency and in opposite directions. If either disk has a fixed frequency of vibration it will behave as the simple system shown in Fig. 2·7. There will then be one point on the shaft in Fig. 4·1 which will not move. This point is called the nodal point. If we assume that L_1 is the distance from this point to disk I_1 and L_2 is the distance to I_2, we can write that

$$L = L_1 + L_2 \tag{4·1}$$

Also since this point stands still, the shaft section L_1 and disk I_1 will behave exactly as the simple system shown in Fig. 2·7. The natural frequency of this system will then be (see equation 2·39)

$$f = \frac{1}{2\pi}\sqrt{\frac{k_{t1}}{I_1}} \tag{4·2}$$

The natural frequency of the system to the right of the node will be

$$f = \frac{1}{2\pi}\sqrt{\frac{k_{t2}}{I_2}} \tag{4·3}$$

Since these frequencies must be equal, we can write that

$$\frac{1}{2\pi}\sqrt{\frac{k_{t1}}{I_1}} = \frac{1}{2\pi}\sqrt{\frac{k_{t2}}{I_2}}$$

or

$$\frac{I_2}{I_1} = \frac{k_{t2}}{k_{t1}} \tag{4·4}$$

Since the spring constant for a circular shaft can be written in the general form $k_t = \dfrac{\pi d^4 G}{32L} = \dfrac{C}{L}$, where C is a constant for a uniform shaft, we can write that

$$k_{t1} = \frac{C}{L_1}$$

$$k_{t2} = \frac{C}{L_2}$$

Upon substituting these values in the equation above we have

$$\frac{I_2}{I_1} = \frac{L_1}{L_2} \tag{4·5}$$

Substituting in equation 4·1 and simplifying, we would have

$$L_1 = \frac{I_2 L}{I_1 + I_2} \tag{4·6}$$

$$L_2 = \frac{I_1 L}{I_1 + I_2} \tag{4·7}$$

giving

$$k_{t2} = \frac{\pi d^4 G}{32 L_2} = \frac{\pi d^4 G (I_1 + I_2)}{32 I_1 L} \tag{4·8}$$

Since the frequency of both sections is the same we can write that

$$f = \frac{1}{2\pi}\sqrt{\frac{\pi d^4 G (I_1 + I_2)}{32 I_1 I_2 L}} \tag{4·9}$$

Illustrative Problem. If $I_1 = 4000$ lb-in.-sec², $I_2 = 1000$ lb-in.-sec², the length of the steel shaft is 20 in., and the diameter 2 in., determine the natural frequency of the system.

Solution. From equation 4·9 we find that the frequency is given by

$$\begin{aligned} f &= \frac{1}{2\pi}\sqrt{\frac{\pi d^4 G (I_1 + I_2)}{32 I_1 I_2 L}} \\ &= \frac{1}{2\pi}\sqrt{\frac{\pi (2^4) 12 (10^6)(4000 + 1000)}{32(4000)1000(20)}} \\ &= 5.46 \text{ cycles/sec} \end{aligned}$$

4·3. Three-Disk Two-Shaft Problem. Figure 4·2 shows three disks having moments of inertia I_1, I_2, and I_3 connected by two shafts having elastic constants k_{t1} and k_{t2}. If β_1 represents the angular position of disk I_1, β_2 of disk I_2, and β_3 of disk I_3, an expression for the equilibrium of each disk can be written. Since damping is neglected we can write that the sum of the torque on disk 1 and the inertia force must be a constant. The torque on the disk will be given by the spring constant times the angle of twist or $k_{t1}(\beta_1 - \beta_2)$, where k_{t1} is the spring constant for shaft 1. The expression of equilibrium for disk 1 would be

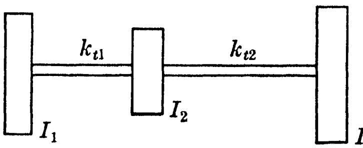


FIG. 4·2.

$$I_1 \frac{d^2\beta_1}{dt^2} + k_{t1}(\beta_1 - \beta_2) = 0 \tag{4·10}$$

A similar expression can be written for disk 2. For this disk the torque from both shafts 1 and 2 must be taken into account. The torque on disk 2 exerted by shaft 1 will be opposite that exerted by shaft 1 on disk 1; it will be $-k_{t1}(\beta_1 - \beta_2)$. The torque exerted by shaft 2 on disk 2 will be $k_{t2}(\beta_2 - \beta_3)$. The sum of these torques plus the inertia torque of the disk must be zero. This expressed analytically would give

$$I_2 \frac{d^2\beta_2}{dt^2} + k_{t2}(\beta_2 - \beta_3) - k_{t1}(\beta_1 - \beta_2) = 0 \qquad [4\cdot 11]$$

A similar expression for disk 3 would give

$$I_3 \frac{d^2\beta_3}{dt^2} - k_{t2}(\beta_2 - \beta_3) = 0 \qquad [4\cdot 12]$$

Adding these, we have

$$I_1 \frac{d^2\beta_1}{dt^2} + I_2 \frac{d^2\beta_2}{dt^2} + I_3 \frac{d^2\beta_3}{dt^2} = 0 \qquad [4\cdot 13]$$

The solution of these differential equations can be obtained by assuming a solution in the form

$$\beta_1 = A_1 \cos (\omega t + \gamma) \qquad [4\cdot 14]$$

$$\beta_2 = A_2 \cos (\omega t + \gamma) \qquad [4\cdot 15]$$

$$\beta_3 = A_3 \cos (\omega t + \gamma) \qquad [4\cdot 16]$$

If these values are substituted in equations 4·10, 4·11, and 4·12, the following expressions are obtained:

$$I_1 A_1 \omega^2 - k_{t1}(A_1 - A_2) = 0 \qquad [4\cdot 17]$$

$$I_2 A_2 \omega^2 + k_{t1}(A_1 - A_2) - k_{t2}(A_2 - A_3) = 0 \qquad [4\cdot 18]$$

$$I_3 A_3 \omega^2 + k_{t2}(A_2 - A_3) = 0 \qquad [4\cdot 19]$$

Upon adding these equations we find

$$I_1 A_1 + I_2 A_2 + I_3 A_3 = 0 \qquad [4\cdot 20]$$

From equations 4·17 and 4·19 we find

$$A_1 = -\frac{k_{t1} A_2}{I_1 \omega^2 - k_{t1}} \qquad [4\cdot 21]$$

$$A_3 = -\frac{k_{t2} A_2}{I_3 \omega^2 - k_{t2}} \qquad [4\cdot 22]$$

If equations 4·21 and 4·22 are substituted in equation 4·20 and the result simplified, we have

$$\frac{I_1I_2I_3}{k_{t1}k_{t2}}\omega^4 - \left[\frac{I_1I_2 + I_1I_3}{k_{t1}} + \frac{I_2I_3 + I_1I_3}{k_{t2}}\right]\omega^2 + (I_1 + I_2 + I_3) = 0 \quad [4\cdot 23]$$

This equation can be solved for ω by first solving for ω^2.

Illustrative Problem. Determine the natural frequencies for a three-mass system, where

$$I_1 = 45{,}700 \text{ lb-in.-sec}^2$$
$$I_2 = 1180 \text{ lb-in.-sec}^2$$
$$I_3 = 30{,}200 \text{ lb-in.-sec}^2$$
$$k_{t1} = 3.45(10^7) \text{ in.-lb/radian}$$
$$k_{t2} = 5.78(10^8) \text{ in.-lb/radian}$$

Solution. From equation 4·23 we would have

$$\frac{45{,}700(1180)30{,}200}{3.45(10^7)5.78(10^8)}\omega^4 - \left[\frac{45{,}700(1180) + 45{,}700(30{,}200)}{3.45(10^7)} + \frac{1180(30{,}200) + 45{,}700(30{,}200)}{5.78(10^8)}\right]\omega^2 + 45{,}700 + 1180 + 30{,}200 = 0$$

$$0.82(10^{-4})\omega^4 - 44.0\omega^2 + 77{,}080 = 0$$

or

$$\omega^4 - 53.6(10^4)\omega^2 + 9.4(10^8) = 0$$

$$\omega^2 = \frac{53.6(10^4) \pm \sqrt{[53.6(10^4)]^2 - 37.6(10^8)}}{2}$$

$$= \frac{53.6(10^4) \pm 10^4\sqrt{53.6^2 - 37.6}}{2}$$

$$\omega^2 = 53.4(10^4) \text{ and } 20(10^2)$$

$$\omega_1 = 731 \text{ radians/sec}$$

$$\omega_2 = 44.7 \text{ radians/sec}$$

$$f_1 = \frac{1}{2\pi}\omega_1 = \frac{7.31(10^2)}{2\pi} = 116 \text{ cycles/sec or } 6960 \text{ cycles/min}$$

$$f_2 = \frac{1}{2\pi}\omega_2 = \frac{44.7}{2\pi} = 7.11 \text{ cycles/sec or } 427 \text{ cycles/min}$$

4·4. Two-Mass Two-Spring System. One type of system frequently encountered in connection with vibration absorption is shown in Fig. 4·3. In this particular case it is assumed that a forced vibration $F \sin \nu t$ is applied to weight W_1. The displacement of weight W_1 from its equilibrium position is designated by x_1; likewise, x_2 represents the displacement of the weight W_2 from its equilibrium position. By writing the equation of equilibrium for the weight W_2, we should have upon simplification

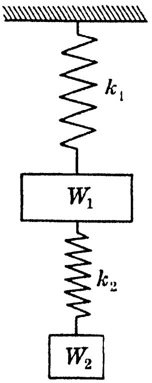


FIG. 4·3.

$$\frac{W_2}{g}\frac{d^2x_2}{dt^2} + k_2(x_2 - x_1) = 0 \qquad [4\cdot 24a]$$

A similar equation can be written for the weight W_1, giving

$$\frac{W_1}{g}\frac{d^2x_1}{dt^2} + k_1x_1 + k_2(x_1 - x_2) = F \sin \nu t \qquad [4\cdot 24b]$$

The equation for the natural frequencies for this system can be determined as in the previous section. It is

$$\omega^4 - \omega^2\left[\frac{(k_1 + k_2)}{W_1} g + \frac{k_2 g}{W_2}\right] + \frac{k_1 k_2 g^2}{W_1 W_2} = 0 \qquad [4\cdot 25]$$

The positive values of ω, which are roots of this equation, are the natural frequencies. Thus, with no damping there are two frequencies at which the amplitude becomes infinite. Figure 5·19 illustrates the amplitude of W_1 for one set of conditions.

It was shown in the previous chapter that when there is only a very small amount of damping the transient part of the general solution eventually approaches zero, leaving only the steady state term which is the particular integral.

It can be shown that the steady state solution for displacement will be obtained by substituting

$$x_1 = x_{01} \sin \nu t$$

$$x_2 = x_{02} \sin \nu t$$

in equations 4·24*a* and 4·24*b*. When reduced and solved simultaneously, the amplitudes become

$$x_{01} = \frac{F\left(k_2 - \frac{W_2}{g}\nu^2\right)}{\left(k_1 + k_2 - \frac{W_1}{g}\nu^2\right)\left(k_2 - \frac{W_2}{g}\nu^2\right) - k_2^2} \qquad [4\cdot 26]$$

or

$$x_{01} = \frac{F}{k_1} \frac{\left[1 - \dfrac{\nu^2}{\omega_{n_2}{}^2}\right]}{\left[1 + \dfrac{k_2}{k_1} - \dfrac{\nu^2}{\omega_{n_1}{}^2}\right]\left[1 - \dfrac{\nu^2}{\omega_{n_2}{}^2}\right] - \dfrac{k_2}{k_1}} \quad [4 \cdot 27]$$

and

$$x_{02} = \frac{F}{k_1} \frac{1}{\left[1 + \dfrac{k_2}{k_1} - \dfrac{\nu^2}{\omega_{n_1}{}^2}\right]\left[1 - \dfrac{\nu^2}{\omega_{n_2}{}^2}\right] - \dfrac{k_2}{k_1}} \quad [4 \cdot 28]$$

where $\omega_{n_1} = \sqrt{\dfrac{k_1 g}{W_1}}$ and $\omega_{n_2} = \sqrt{\dfrac{k_2 g}{W_2}}$.

The applications of these equations are discussed in Chapter V, under vibration absorbers.

4·5. Three-Mass Two-Spring System. The three-mass system as shown in Fig. 4·4 corresponds to the torsional system shown in Fig. 4·2. The solution can be obtained by following the method outlined in sections 4·2 and 4·4. The natural frequency can be calculated from the following equation which is similar to 4·23:

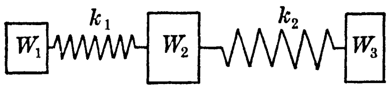


Fig. 4·4.

$$\frac{W_1 W_2 W_3}{g^3 k_1 k_2} \omega^4 - \left[\frac{W_1 W_2 + W_1 W_3}{g^2 k_1} + \frac{W_2 W_3 + W_1 W_3}{g^2 k_2}\right] \omega^2 + \frac{W_1 + W_2 + W_3}{g} = 0 \quad [4 \cdot 29]$$

Systems of still higher degrees of freedom can be solved in a similar manner. As can be seen from comparison of equations 4·9 and 4·23, the addition of another degree of freedom increases the difficulty of solution. Although equations can be derived by these methods for such cases, the mobility method given in Chapter VII is to be preferred, and for systems of several degrees of freedom reference should be made to this method.

4·6. Tabulation Method for Torsional Vibrations. In analyzing torsional vibrations it is usually convenient to reduce the system to an equivalent shaft which has the same ability to store potential energy as the original system. At the same time all the masses are reduced to

equivalent disks that have the same mass effects as the original system. This is discussed in detail in Chapter VI. The resulting equivalent system consists of several disks mounted on a shaft, such as is shown in Fig. 4·5. Several convenient, practical methods have been proposed

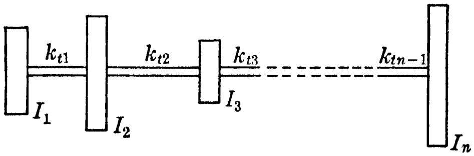


FIG. 4·5.

for determining the natural torsional frequencies of such systems. A tabulation method proposed by Holzer[16] is often used to obtain the natural frequencies and the relative elastic curve.

The equations for a torsional system may be determined from Newton's laws of motion so that at the natural frequency the elastic torque and inertia torques are in equilibrium. This can be done either by the method given in section 4·3 or by a direct application of the condition for equilibrium

$$\Sigma T = I\alpha$$

When a system such as is shown in Fig. 4·5 is vibrating freely, no external periodic torque need be applied to maintain the vibration. Thus the disks move relative to one another with relative inertia torques applied between disks, but the total of all these internal torques must be zero, giving

$$\Sigma I\alpha = I_1\alpha_1 + I_2\alpha_2 + \cdots + I_n\alpha_n = 0 \qquad [4\cdot30]$$

The angular acceleration is given as

$$\alpha = \beta\omega^2$$

where β = maximum relative angle in radians
ω = frequency in radians per second

This reduces equation 4·30 to

$$(I_1\beta_1 + I_2\beta_2 + \cdots + I_n\beta_n)\omega^2 = 0 \qquad [4\cdot31]$$

To solve this equation it is necessary to obtain some relationship between the maximum angles of vibration. To do this, the angles of motion may be measured in terms of one particular disk. It is convenient to start analyzing from one of the end disks. In this case assume that all angles are measured in terms of β_1. Now disk I_1 causes

a maximum torque $I_1\omega^2\beta_1$. As this is the only torque applied to shaft 1 the angular deflection in this shaft is $I_1\omega^2\beta_1/k_{t1}$. Thus

$$\beta_2 = \beta_1 - \frac{I_1\omega^2\beta_1}{k_{t1}} \qquad [4\cdot32]$$

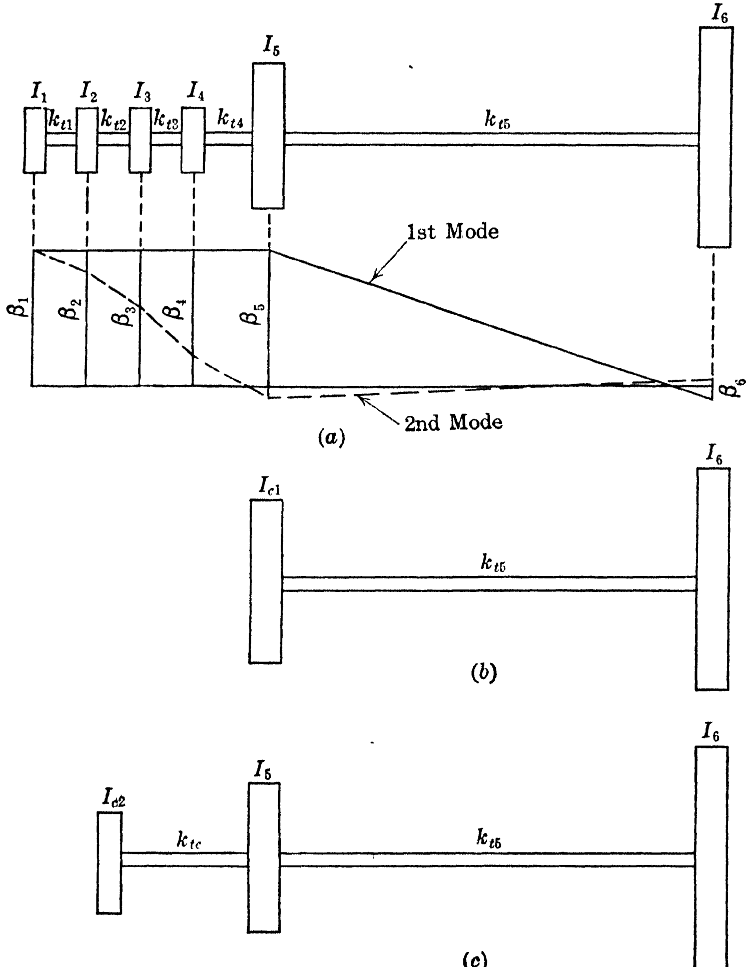


FIG. 4·6.

The torque acting on shaft 2 is $I_1\omega^2\beta_1 + I_2\omega^2\beta_2$ so that the angle of twist in shaft 2 is $(I_1\beta_1 + I_2\beta_2)\dfrac{\omega^2}{k_{t2}}$ or

$$\beta_3 = \beta_2 - \frac{\omega^2}{k_{t2}}(I_1\beta_1 + I_2\beta_2) \qquad [4\cdot33]$$

This analysis may be continued until

$$\beta_n = \beta_{n-1} - \frac{\omega^2}{k_{tn-1}} (I_1\beta_1 + I_2\beta_2 + \cdots + I_{n-1}\beta_{n-1}) \qquad [4\cdot 34]$$

These values for the angles can then be reduced to some multiple or fraction of β_1. If these reduced values are substituted in equation 4·31, β_1 can be factored out and, since it is not zero, the equation can be divided by it, thereby eliminating β_1. It will, however, be necessary to leave β_1 in the equation and take it as being one radian. This reduces the problem to solving an algebraic equation of higher degree. It is most convenient to solve problems having more than four masses by making successive approximations. This leads to the use of the Holzer tabulation method. With this method it is convenient to set up a table that is divided into eight columns, such as Table 4·1. The values given for the system shown in Fig. 4·6 are listed in Table 4·1.

TABLE 4·1

HOLZER TABULATION FOR TORSIONAL SYSTEMS

$\omega = 2335$ $\qquad \omega^2 = 5{,}450{,}000$

1	2	3	4	5	6	7	8
Item	I	$I\omega^2$	β	$I\omega^2\beta$	$\Sigma I\omega^2\beta$	k_t	$\frac{1}{k_t}\Sigma I\omega^2\beta$
1	0.18	0.981(10^6)	1	0.981(10^6)	0.981(10^6)	7.02 (10^6)	0.140
2	0.18	"	0.860	0.844 "	1.825 "	6.11 "	0.299
3	0.18	"	0.561	0.550 "	2.375 "	7.02 "	0.338
4	0.18	"	0.223	0.219 "	2.594 "	8.08 "	0.321
5	4.96	27.0(10^6)	−0.098	−2.645 "	−0.051 "	0.0177 "	−2.88
6	91.6	499(10^6)	+2.78	1390 "	1390 "		

The first column gives the item number. The second column lists the moment of inertia of the masses as numbered in the first column. A value of the frequency in radians per second must be estimated, and then column 3 can be determined by multiplying I by ω^2. Column 4 starts out with the value $\beta_1 = 1$ radian, but after that it is equal to the

preceding value of β minus the corresponding value of column 8, which gives the relative angle. After column 5 has been calculated all the existing values of column 5 are added to give the value of column 6, which is the numerator in the last term of equation 4·34. Column 7 lists the values of the torsional spring constant for the shafts in the same order as the masses are numbered. Column 8 may now be determined by dividing column 6 by the value of the spring constant in column 7. It represents the relative angle between the two masses. This term in column 8, when subtracted from the value in column 4, gives the angular position of the next disk. The values are then found for this disk, then the next, and so on, until the last value of column 6 has been determined. Thus the relative angles may be determined, and the torque on each shaft may be evaluated.

If the value of ω assumed at the beginning is a natural frequency, the last term in column 6 should be 0 to satisfy equation 4·31. This was shown to be necessary in order that the moment of momentum for the system should remain constant. For practical purposes the value of the last term need be only near 0. If it is not near 0, another value of ω must be assumed. In the illustrative table the value is very large. Actually it is relatively close to 0, for a change of ω to 2330 brings the total in column 6 very close to 0. By plotting the values of ω and the last term in column 6 from several trials, it is possible to determine more rapidly where the last term will be zero. This process may be repeated until all the natural frequencies have been determined. The first natural frequency is found when the values of β pass through 0 only once, the second when it passes through twice, the third three times, etc. The values of β may be plotted at their corresponding points along the shaft as shown in Fig. 4·6 to give the elastic curve. This indicates the relative movement of the disks or parts of the system. The node will be at the point where the elastic curve crosses the axis. One frequency will always be zero, but this has no interpretation other than that all disks rotate together with no relative motion. Counting zero as a natural frequency in a torsional system such as this, the number of natural frequencies will be equal to the number of masses.

Without an approximate answer to start with, the solution of a multidisk and shaft problem becomes very difficult. Approximations, therefore, are made that will enable one to try reasonable values for the frequency. This is done by grouping several disks together. In Fig. 4·6, for instance, I_1, I_2, I_3, I_4, and I_5 are added together to give one mass. Since I_5 is much larger than any of the others, the shaft extends approximately to this disk, as shown in Fig. 4.6*b*. This gives a two-mass system which will yield the first natural frequency. Another ap-

proximation could be made by combining I_1, I_2, I_3, and I_4 into one disk located between I_2 and I_3. This system would be a three-mass system as shown in Fig. 4·6c and could be easily solved. Since I_6 is so large compared to the rest of the system and is connected to the rest of the system by a relatively long shaft, the combined mass and I_5 might be considered alone as a two-mass sytem. The values obtained by these methods may now be used as a starting point for the tabulation method.

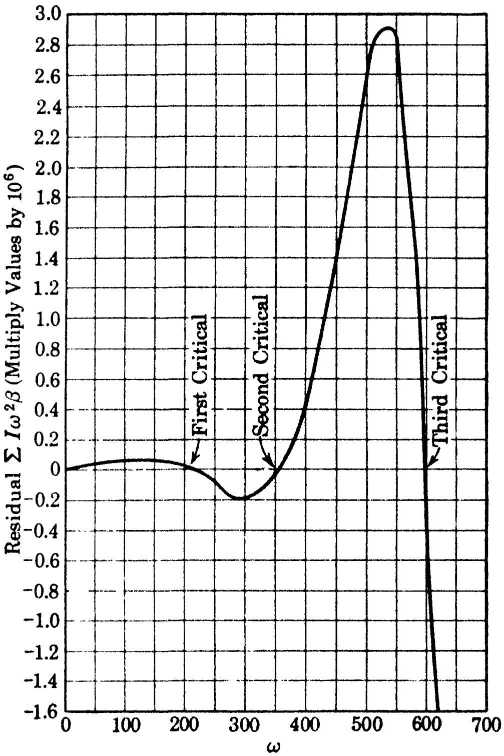


FIG. 4·7.

4·7. Application of Tabulation Method. Facility in use of the tabulation method can only be obtained through practice and experience. The best way to arrive at a starting value is through approximations as outlined in the previous section. Once starting values have been selected a plot of residual torque or the final value of $\Sigma I\omega^2\beta$ can be used as a guide to selection of new trial values. Figure 4·7 shows such a plot. Generally a different scale is needed for each frequency since the value of the residual torque increases quite rapidly with the frequency. Figure 4·8

shows a plot of residual torque versus ω for the second critical speed indicated in Figure 4·7. In both cases it can be seen that a small change in frequency in the vicinity of the natural frequency results in a large variation in the magnitude of the residual torque.

Another type of curve helpful in checking results and in obtaining a better understanding of vibration systems is shown in Figure 4·9. These are the elastic curves for the three natural frequencies. They are

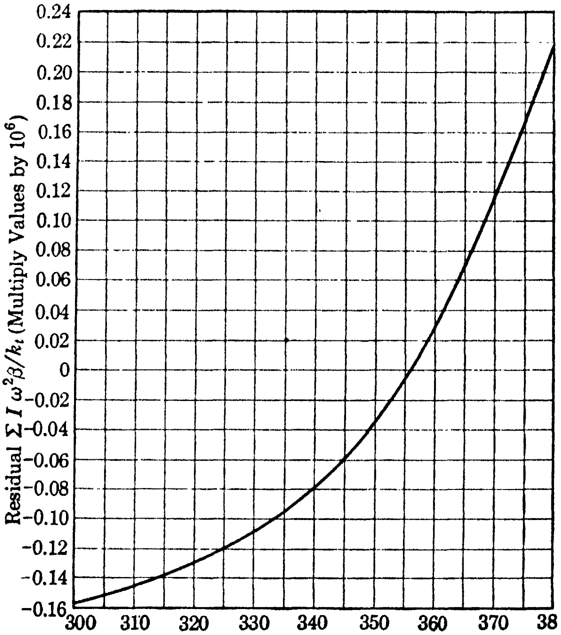


Fig. 4·8.

a plot of the angle β for the various masses at the natural frequency. Figure 4·9*a* shows that the first mode of vibration, or lowest natural frequency, occurs with masses I_1 and I_2 vibrating in one direction and masses I_3 and I_4 in the other. The second mode or second natural frequency has a normal curve as shown in Fig. 4·9*b*. In this case masses I_1 and I_4 vibrate together and oppose I_2 and I_3. In the third mode, as shown in Fig. 4·9*c*, there is a node between each mass.

The calculations used in obtaining these curves are the same as described in the previous section. Table 4·2 shows a calculation for the second natural frequency which is too low. Table 4·3 shows a calculation giving a value very close to the natural frequency and Table 4·4 a calculation above the natural frequency. By plotting values of residual

torque for such calculations against ω as was done in Fig. 4·8, a reasonably accurate value can usually be found with only two or three calculations. If a normal curve is desired, a calculation has to be made at a value very close to the correct ω to give the true value of β.

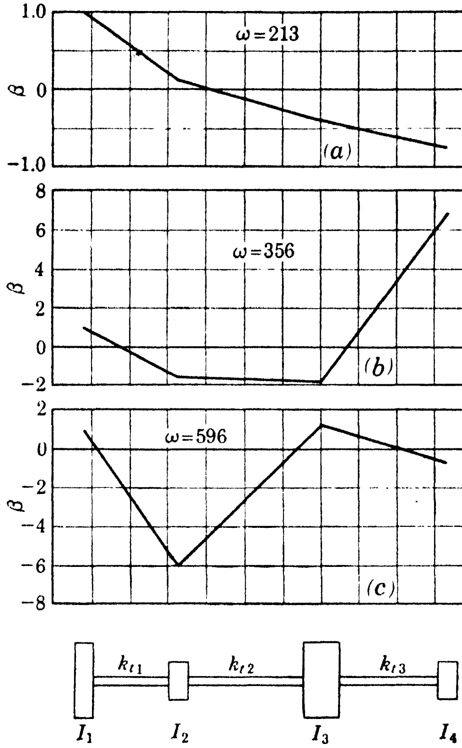


Fig. 4·9.

TABLE 4·2

$\omega = 600$ $\qquad \omega^2 = 0.36(10^6)$

1	2	3	4	5	6	7	8
Item	I	$I\omega^2$	β	$I\omega^2\beta$	$\Sigma I\omega^2\beta$	k_t	$\frac{1}{k_t}\Sigma I\omega^2\beta$
1	2	$0.72(10^6)$	1.00	$0.72(10^6)$	$0.72(10^6)$	$0.1(10^6)$	7.2
2	1	$0.36(10^6)$	−6.2	$-2.23(10^6)$	$-1.51(10^6)$	$0.2(10^6)$	−7.55
3	4	$1.44(10^6)$	+1.35	$+1.94(10^6)$	$0.43(10^6)$	$0.1(10^6)$	4.3
4	1	$0.36(10^6)$	−2.95	$-1.06(10^6)$	$-0.63(10^6)$		

TABLE 4·3

$\omega = 596$ $\omega^2 = 0.355(10^6)$

1	2	3	4	5	6	7	8
Item	I	$I\omega^2$	β	$I\omega^2\beta$	$\Sigma I\omega^2\beta$	k_t	$\frac{1}{k_t}\Sigma I\omega^2\beta$
1	2	$0.710(10^6)$	1.00	$0.710(10^6)$	$0.710(10^6)$	$0.1(10^6)$	
2	1	$0.355(10^6)$	−6.10	$-2.162(10^6)$	$-1.452(10^6)$	$0.2(10^6)$	
3	4	$1.42(10^6)$	+1.16	$+1.645(10^6)$	$+0.193(10^6)$	$0.1(10^6)$	
4	1	$0.355(10^6)$	−0.77	$-0.280(10^6)$	$-0.087(10^6)$		

TABLE 4·4

$\omega = 610$ $\omega^2 = 0.371(10^6)$

1	2	3	4	5	6	7	8
Item	I	$I\omega^2$	β	$I\omega^2\beta$	$\Sigma I\omega^2\beta$	k_t	$\frac{1}{k_t}\Sigma I\omega^2\beta$
1	2	$0.742(10^6)$	1.0	$0.742(10^6)$	$0.742(10^6)$	$0.1(10^6)$	7.42
2	1	$0.371(10^6)$	−6.42	$-2.382(10^6)$	$-1.64(10^6)$	$0.2(10^6)$	−8.19
3	4	$1.77(10^6)$	1.77	$2.63(10^6)$	$1.00(10^6)$	$0.1(10^6)$	10.0
4	1	$-8.23(10^6)$	−8.23	$-3.05(10^6)$	$-2.05(10^6)$		

4·8. Tabulation Method for Linear Vibrations. The tabulation method used for torsional systems can be easily adapted to such linear systems as are shown in Fig. 4·10. Once displaced no external forces are

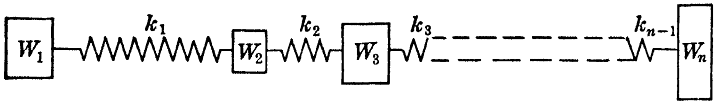


FIG. 4.10.

needed to maintain the vibrations, and so the sum of the inertia forces must be zero. Therefore

$$\frac{W_1}{g}a_1 + \frac{W_2}{g}a_2 + \cdots + \frac{W_n}{g}a_n = 0$$

This reduces to

$$\left[\frac{W_1}{g}x_1 + \frac{W_2}{g}x_2 + \cdots + \frac{W_n}{g}x_n\right]\omega^2 = 0 \qquad [4\cdot35]$$

The displacements of all the masses can be measured in terms of the displacement of the first mass as was done in the torsional system. A table, therefore, may be made to solve for the frequencies of the system. Dudley uses this method to solve for the frequencies of longitudinal vibration in a train of cars.[17] One trial value for a train of four cars and an engine is given in Table 4·5.

TABLE 4·5

HOLZER TABULATION FOR LINEAR SYSTEMS

$\omega = 8.79$ $\qquad$ $\omega^2 = 77.2$

1	2	3	4	5	6	7	8
Item	$\frac{W}{g}$	$\frac{W}{g}\omega^2$	x	$\frac{W}{g}\omega^2 x$	$\sum\frac{W}{g}\omega^2 x$	k	$\frac{\Sigma W\omega^2 x}{gk}$
1	777	60.0 (10^3)	1.00	60.0 (10^3)	60.0(10^3)	150(10^3)	0.400
2	117	9.03 "	0.600	5.4 "	65.4 "	100 "	0.654
3	117	9.03 "	−0.054	−0.49 "	64.9 "	100 "	0.649
4	388	30.0 "	−0.703	−21.1 "	43.8 "	100 "	0.438
5	388	30.0 "	−1.141	−34.2 "	9.6 "		

Columns 1 to 4 have the same significance as those in Table 4·1. Columns 5 and 6 represent the individual and total forces acting on the system. It is necessary for the last term in column 6 to be 0 in order that equation 4·35 will be satisfied. Column 8 represents the displacement between masses. Thus the linear and torsional systems correspond very closely.

PROBLEMS

4·1. Determine the natural frequencies of a two-mass system such as is shown in Fig. P4·1.

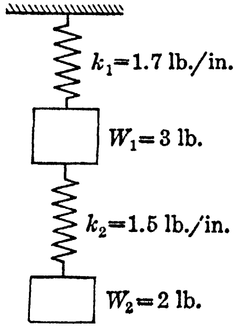


FIG. P4·1.

4·2. A particular system is reduced to a two-mass two-spring system as shown in Fig. 4·3. The first mass weighs 260 lb; the second weighs 185 lb. The first spring has a spring constant $k_1 = 730$ lb/in., and the spring between the masses has a constant $k_2 = 215$ lb/in. What are the natural frequencies?

4·3. In a three-disk system such as is shown in Fig. 4·2, $I_1 = 26$, $I_2 = 38$, and $I_3 = 18$ lb-in.-sec.² Disks I_1 and I_2 are separated by a shaft 2 in. in diameter and 16 in. long, whereas I_2 and I_3 are separated by a shaft $1\frac{1}{2}$ in. in diameter and 12 in. long. What are the natural frequencies of this system?

4·4. A disk with a moment of inertia $I_1 = 4.8$ lb-in.-sec² is connected by a stepped shaft to another disk with a moment of inertia $I_2 = 2.8$ lb-in.-sec². The larger portion of the shaft is 1 in. in diameter and 14 in. long and is connected to disk I_1. The remaining portion of the shaft is $\frac{3}{4}$ in. in diameter and 6 in. long. What is the natural frequency of this system?

4·5. A system reduces to a two-disk system as shown in Fig. 4·1 with $I_1 = 12$ lb-in.-sec² and $I_2 = 4.2$ lb-in.-sec.² The unit diameter steel shaft connecting the two disks is 6.8 in. long. What is the natural frequency of this system?

4·6. The torsional vibrations in a radial engine can be found by reducing the propeller, connecting rods and pistons, and supercharger to equivalent disks by methods explained in Chapter VI. For one particular airplane the equivalent disk and shaft system is shown in Fig. P4·6. $I_1 = 90$, $I_2 = 12$, $I_3 = 4$ lb-in.-sec². The elasticity of the shaft between the propeller I_1 and the crank I_2 is $k_{t1} = 6 \times 10^6$ lb-in./radian, whereas for the other shaft $k_{t2} = 0.4 \times 10^6$. What are the natural frequencies for torsional vibration?

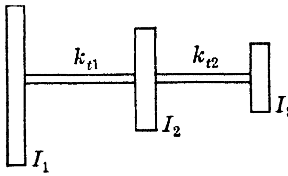


FIG. P4·6.

4·7. Show how equation 4·29 for a three-mass and two-spring system may be reduced to equation 4·25 for a two-mass and two-spring system.

4·8. Three weights on a frictionless support are separated by two springs. The weights are $W_1 = 25$ lb, $W_2 = 13$ lb, and $W_3 = 32$ lb, and the spring constants are $k_1 = 245$ lb/in. and $k_2 = 160$ lb/in. What are the natural frequencies for this system?

4·9. The elevators on a plane are rigidly connected by a hollow aluminum tube. This system can be approximated by a two-disk and shaft system. If $I_1 = I_2 = 5.2$ lb-in.-sec² and the tube has a polar moment of inertia of 0.038 in.⁴ and is 70 in. long, determine the natural frequency of the system in cycles per minute. (Use $G = 3{,}800{,}000$.)

4·10. Draw a curve for residual torque for the region from zero to 10,000 cycles/min.

4·11. Draw the normal curves for the two natural frequencies of problem 4·6.

4·12. Derive equation 4·25 starting with equilibrium equations 4·24*a* and 4·24*b*.

4·13. Write the equilibrium equations applying to Fig. 4·4, and derive equation 4·29.

4·14. A small airplane engine and propeller system may be reduced to the system shown in Fig. P4·14, where $I_1 = 10$, $I_2 = I_3 = I_4 = I_5 = 0.10$ lb-in.-sec^2, and $k_{t1} = 1.3 \times 10^6$, $k_{t2} = k_{t3} = k_{t4} = 1.5 \times 10^6$ lb-in./radian. What is the fundamental frequency for the system? What is the second frequency?

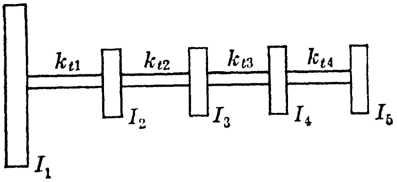


FIG. P4·14.

4·15. A Diesel-driven motorship drive reduces to the equivalent system of disks shown in Fig. P4·15, where

$$I_1 = I_3 = I_4 = I_6 = 4000 \text{ lb-in.-sec}^2$$
$$I_2 = I_5 = 3000 \text{ lb-in.-sec}^2$$
$$I_7 = 85{,}000,\ I_8 = 92{,}000 \text{ lb-in.-sec}^2$$
$$k_{t1} = k_{t2} = k_{t4} = k_{t5} = 660 \times 10^6 \text{ lb-in./radian}$$
$$k_{t3} = 410 \times 10^6 \text{ lb-in./radian}$$
$$k_{t6} = 470 \times 10^6 \text{ lb-in./radian}$$
$$k_{t7} = 14.8 \times 10^6 \text{ lb-in./radian}$$

What are the first two natural frequencies for this torsional system?

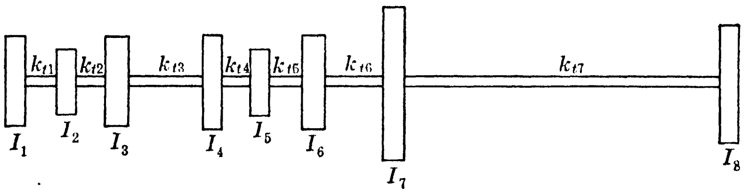


FIG. P4·15.

4·16. A Diesel motor-generator is reduced to an equivalent system as shown in Fig. P4·16, where

$$I_1 = I_2 = I_3 = I_4 = I_5 = I_6 = 3600 \text{ lb-in.-sec}^2$$
$$I_7 = 92{,}000 \text{ lb-in.-sec}^2$$
$$I_8 = 32{,}000 \text{ lb-in.-sec}^2$$
$$k_{t1} = k_{t2} = k_{t4} = k_{t5} = 820 \times 10^6 \text{ lb-in./radian}$$
$$k_{t3} = 610 \times 10^6 \text{ lb-in./radian}$$
$$k_{t6} = 586 \times 10^6 \text{ lb-in./radian}$$
$$k_{t7} = 11.6 \times 10^6 \text{ lb-in./radian}$$

(*a*) What is the lowest natural frequency?
(*b*) What is the second critical speed?
(*c*) The third?

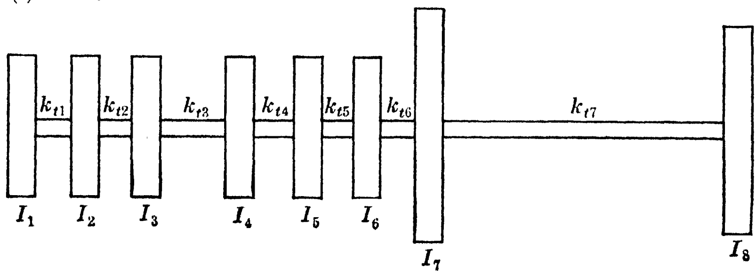


FIG. P4·16.

4·17. One typical oil well pumping system reduces to an equivalent shaft and disk problem as shown in Fig. P4·17. The equivalent steel shaft is 1 in. in diameter and of the lengths given. Find the natural frequencies by approximating with a two-mass system. With a three-mass system.

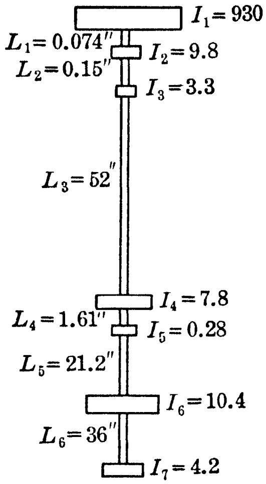


FIG. P4·17.

4·18. What are the first and second natural frequencies in problem 4·17, determined by using the tabulation method?

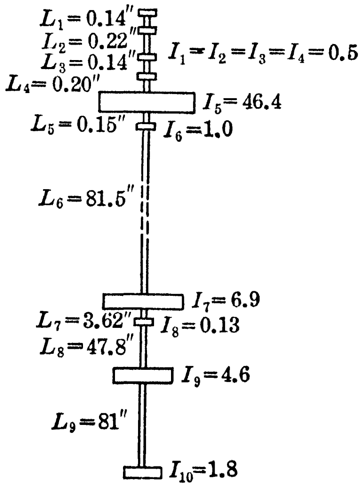


FIG. P4·19.

4·19. A multicylinder engine drives an oil pumping unit and can be reduced to a disk and shaft system as shown in Fig. P4·19. The steel shaft is 1 in. in diameter. What are the first, second, and third natural frequencies?

4·20. A locomotive weighing 620,000 lb is pulling four box cars. The first car has a gross weight W_2 = 110,000 lb, the second and third weigh 170,000 lb each, and the last one weighs 75,000 lb. The spring constant of the coupling between the locomotive and the first car is 190,000 lb/in., and the remaining couplings have elasticities equal to 130,000 lb/in. What are the first and second natural frequencies for this system?

4·21. A refrigerating unit has the dimensions indicated in Fig. P4·21. The motor compressor units are essentially similar in mass characteristics. Each weighs 40 lb

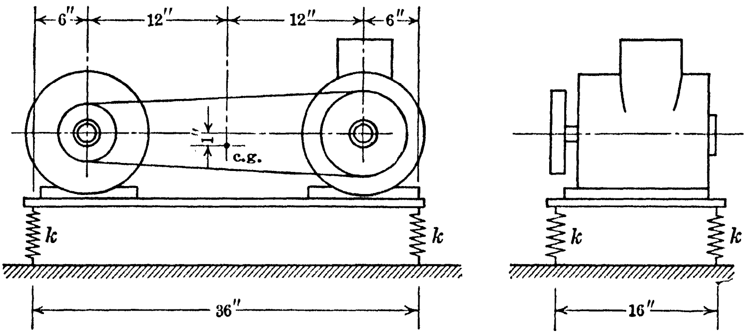


FIG. P4·21.

and has a moment of inertia approximately equal to a cylinder 8 in. in diameter and 12 in. long. The base weighs 20 lb, and each of the four springs has a spring constant k = 50 lb/in. Make whatever assumptions and approximations are necessary and reasonable, and then solve for the vertical frequencies and the two torsional rocking frequencies. How might the frequencies be made more nearly equal?

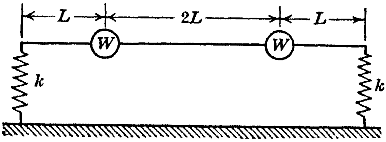


FIG. P4·22.

4·22. (*a*) Set up the differential equations necessary to solve for the natural frequencies and modes or different forms of vibration for the system represented in Fig. P4·22.

(*b*) Solve for the frequencies.
(*c*) Solve for the modes of vibration.

CHAPTER V

VIBRATION ISOLATION AND ABSORPTION

5·1. Introduction. Since all moving machinery has some degree of unbalance, machines will set up forces which may be transmitted to the supporting structure. These forces will produce vibrations in the machine and in surrounding bodies which may be manifested as noise or actual movement. Whether or not such noise or movement is objectionable will depend on the conditions under which it exists, as some equipment or people are more susceptible to vibration than others. Continuous noise and vibration cause fatigue and lower the efficiency of the worker as well as reduce the efficiency of the machine since it must supply the energy of vibration. Claims have been made of increases in operation efficiency of personnel and machines as a result of vibration elimination. In the operation of some types of equipment such as instruments it is necessary to eliminate the vibration to insure proper operation of the equipment.

The vibrations mentioned in the previous paragraph may be divided into those traveling by radiation and those traveling by conduction. Radiating vibrations embrace sound and noise primarily. Conductive vibrations are transmitted by contact between bodies. There are, in general, four ways to reduce the effects of vibration. The first is by elastic suspension. Springs, rubber, cork, etc., are placed between the machine and its supporting structure to reduce the effect of the vibration. The second is by using dynamic vibration absorbers in which an additional elastic member with a mass is added to the vibrating system to absorb the energy that would otherwise be transmitted to the supporting members. The third type of vibration prevention is the use of filters or screens such as mufflers which are employed mostly to reduce noise and sounds. The fourth is by removing the source of the vibration. This often involves a detailed experimental and mathematical investigation both to determine the cause and magnitudes of the vibration and to investigate methods of its elimination.

5·2. Elastic Suspension of Simple Undamped Mass. The purpose of of an elastic suspension is to isolate the vibrating machine from the supporting structure. A system consisting of a weight on a spring and acted upon by a disturbing force (see Fig. 5·1) will be considered. For

simplicity the support is considered rigid. Although damping is small as in helical springs there is still sufficient damping present to make the transient term drop out and leave only the steady state part of the solution as given by equation 3·30. If the disturbing force is F, the component in one direction will be $F \cos \nu t$.

The force transmitted to the substructure is due to the spring's deflection and as such is equal to $kx \cos \nu t$. The usual criterion of the amount of isolation present is termed the transmissibility and is, in effect, the ratio of the force on the substructure to the disturbing force. Therefore, the transmissibility is

$$\text{T.R.} = \frac{kx_0 \cos \nu t}{F \cos \nu t} = \frac{kx_0}{F} \qquad [5\cdot 1]$$

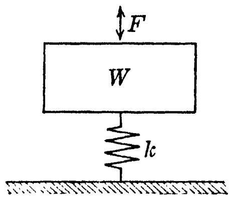


FIG. 5·1.

From equation 3·34 we found that the amplitude of motion, x, was equal to $\frac{F}{k}\left[\frac{1}{1-\left(\frac{\nu}{\omega_n}\right)^2}\right]$ so that the transmissibility is given by the magnification factor

$$\text{T.R.} = \frac{k}{F}\left(\frac{F}{k}\right)\left[\frac{1}{1-\left(\frac{\nu}{\omega_n}\right)^2}\right] = \frac{1}{1-\left(\frac{\nu}{\omega_n}\right)^2} \qquad [5\cdot 2]$$

The same result is obtained for a weight that is to be isolated from the vibrating structure supporting it. The displacement of the weight is given by equation 3·34 as being

$$x = \frac{F}{k}\left[\frac{1}{1-\frac{\nu^2}{\omega_n^{\,2}}}\right]$$

Obviously the displacement varies with the magnification factor so the transmissibility is given by the same factor as before, i.e.,

$$\text{T.R.} = \frac{1}{1-\left(\frac{\nu}{\omega_n}\right)^2}$$

The curve for the transmissibility is the same as for the magnification factor, Fig. 3·12. Actually, however, only the absolute motion is of interest so that all values are plotted as positive, as in Fig. 5·2.

Several conclusions may be drawn from a study of this curve. First of all, at very low ratios of the forced frequency to the natural frequency the force transmitted is approximately equal to the disturbing force.

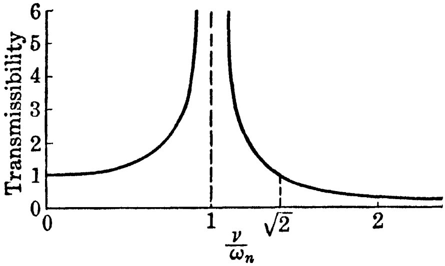


FIG. 5·2.

This would correspond to the case where relatively stiff springs are used, and very little deflection would exist. As the ratio of the impressed frequency approaches the resonant speed, the force actually becomes greater until at the resonant speed it theoretically becomes infinite. Actually enough damping is present to limit the amplitude. As the ratio of speeds is increased the magnitude of force decreases, and at a

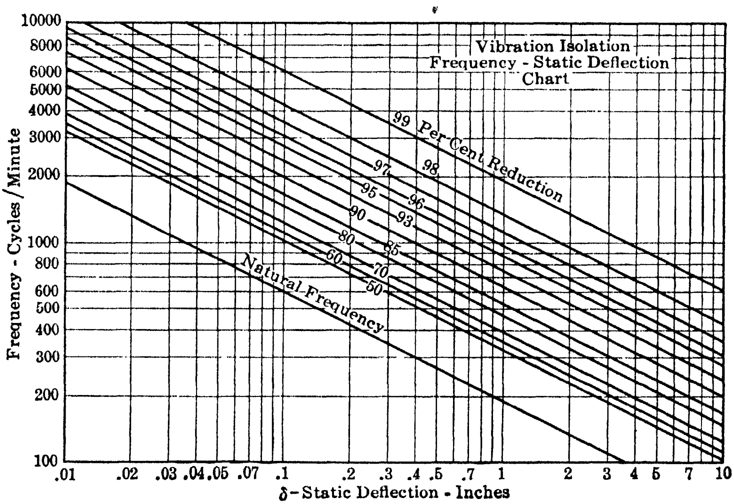


FIG. 5·3.

ratio of $\sqrt{2}:1$ the transmitted force again equals the disturbing force. Beyond this point the transmitted force decreases farther and eventually approaches zero. This last region is, therefore, the most desirable and

corresponds to the case of relatively soft springs which will give a low resonant speed. Equation 5·2 gives a negative transmissibility for values of the speed ratio above 1. The sign of the equation is changed to make this value positive in the region of good isolation so that it can be used directly in solving problems. The equation then becomes

$$\text{T.R.} = \frac{1}{\left(\frac{\nu}{\omega_n}\right)^2 - 1} \qquad [5\cdot 3]$$

The percentage of reduction of vibration may also be determined from the chart given in Fig. 5·3. One line may be used for the natural frequency if the static deflection is known. The remaining lines show what deflection is necessary to give a certain percentage of isolation, that is, (100 − T.R.)100. Usually 80 to 95% isolation is sufficient; more than 98% is seldom required. The forced frequency ν is taken as the lowest disturbing frequency.

Illustrative Problem. A motor weighing 228 lb runs at 1760 rpm. It is to be mounted upon four springs in a test laboratory where it is desirable to get 90% reduction of shaking force. What spring constant should the springs have if they all carry equal loads? What is the natural frequency?

Solution. The transmissibility is 10%. The known values may be substituted in equation 5·3, which then can be solved for the natural frequency. Notice that the speeds may be in any units provided they are the same.

$$\text{T.R.} = \frac{1}{\left(\frac{\nu}{\omega_n}\right)^2 - 1}$$

$$0.10 = \frac{1}{\left(\frac{1760}{N}\right)^2 - 1}$$

$$N = 530 \text{ rpm}$$

The natural frequency is, therefore, 530 rpm. In Chapter II it was shown that the natural frequency depended only upon the static deflection. This means the static deflection can be found by substituting in equation 2·27

$$N = f = \frac{60}{2\pi}\sqrt{\frac{g}{\delta_{st}}}$$

$$\delta_{st} = \left(\frac{60}{2\pi 530}\right)^2 386 = 0.125 \text{ in.}$$

The static deflection must be 0.125 in.; so the spring constant is

$$k = \frac{F}{4}\frac{1}{\delta_{st}} = \frac{228}{4(0.125)} = 456 \text{ lb/in.}$$

The same natural frequency and static deflection might have been found by referring to Fig. 5·3. If the disturbing force were known, the force transmitted to the motor foundation could then be calculated.

5·3 Elastic Suspension of Simple Systems with Damping. Systems cannot be treated as above when considerable damping is present. The transmissibility is again the measure of the forces transmitted, but the force transmitted is the vector sum of the force due to the spring deflection and the force due to the damping element. The damping depends upon the velocity so that it leads the displacement by 90° and is equal to $r\dfrac{dx}{dt} = r\nu x \cos \nu t$. The maximum total force transmitted is

$$\sqrt{(kx)^2 + (r\nu x)^2} = x\sqrt{k^2 + (r\nu)^2}$$

The amplitude x is the same as that for forced vibrations with damping as given by equation 3·32, i.e.,

$$x = \frac{F}{k}\frac{1}{\sqrt{\left[1 - \left(\dfrac{\nu}{\omega_n}\right)^2\right]^2 + \left(2\dfrac{r}{r_c}\dfrac{\nu}{\omega_n}\right)^2}}$$

This leads to the transmissibility expression

$$\text{T.R.} = \frac{x\sqrt{k^2 + (r\nu)^2}}{F} = \frac{F}{kF}\frac{\sqrt{k^2 + (r\nu)^2}}{\sqrt{\left[1 - \left(\dfrac{\nu}{\omega_n}\right)^2\right]^2 + \left(2\dfrac{r}{r_c}\dfrac{\nu}{\omega_n}\right)^2}}$$

$$\text{T.R.} = \sqrt{\frac{1 + \left(2\dfrac{r}{r_c}\dfrac{\nu}{\omega_n}\right)^2}{\left[1 - \left(\dfrac{\nu}{\omega_n}\right)^2\right]^2 + \left(2\dfrac{r}{r_c}\dfrac{\nu}{\omega_n}\right)^2}} \qquad [5\cdot4]$$

This equation reduces to the magnification factor if the damping is set equal to zero. The effect of damping can best be shown by reference to Fig. 5·4. The effect of various amounts of damping are indicated. We can see that below a frequency ratio of $\sqrt{2} : 1$ damping is helpful and particularly at resonant conditions. Above the ratio of $\sqrt{2} : 1$ the damping is undesirable. From a practical viewpoint the damping

may be desirable in this region if there is a possibility of operating at or frequently passing through resonant conditions. The effects of damping in this region are small and may be overcome by using a higher ratio of $\frac{\nu}{\omega_n}$.

Care must be exercised in using equation 5·4 since again the transmissibility is negative when the ratio is above 1 just as in the previous

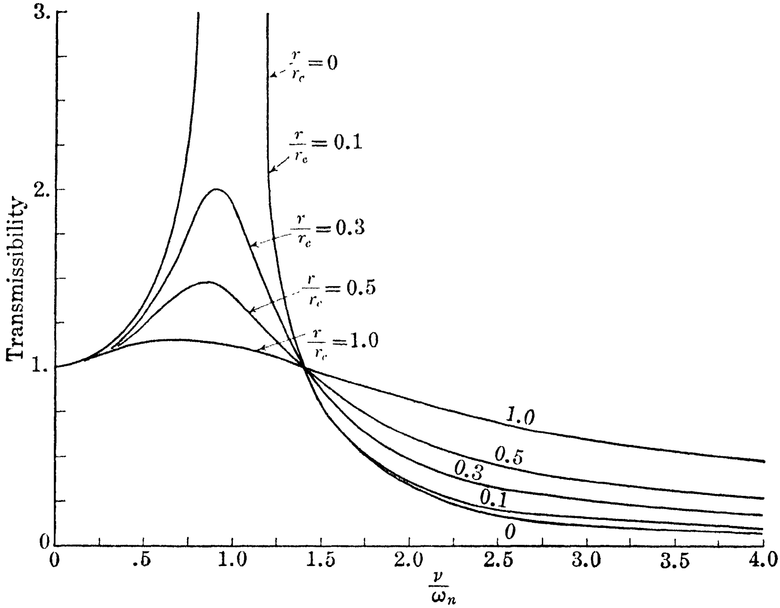


FIG. 5·4.

case. Actually little can be done with the above analysis other than to generalize, for general data on damping factors are practically nonexistent. Specific cases have been considered, but no general data are available. Experience and trial and error must be used.

In some cases excessive amplitudes are prevented by adding auxiliary springs to the system. These springs, however, do not act until the amplitude reaches a definite value. Such springs have the effect of stiffening the system, but the characteristics are no longer linear over the entire range of amplitudes. An analysis of such a system with nonlinear springs is beyond the scope of this book. For discussions of nonlinear springs reference should be made to Timoshenko [18] or Den Hartog.[19]

5·4. Elastic Suspension on Non-rigid Support. The previous discussion has assumed the floor or foundation to be rigid. Many floors

and substructures, however, are not rigid. It is more accurate to consider the system as a two-mass system, as indicated in Fig. 5·5, with the floor as a mass and also as an elastic member. Determination of the mass and elasticity of the floor is difficult. An oscillator may be used to determine the natural frequency of the floor as this is of importance in determining the vibrational characteristics to be used in the two-mass system. In any case, the results of these calculations will not be very accurate. Experience is necessary to know how to handle these cases.

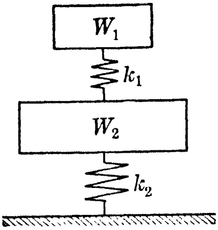
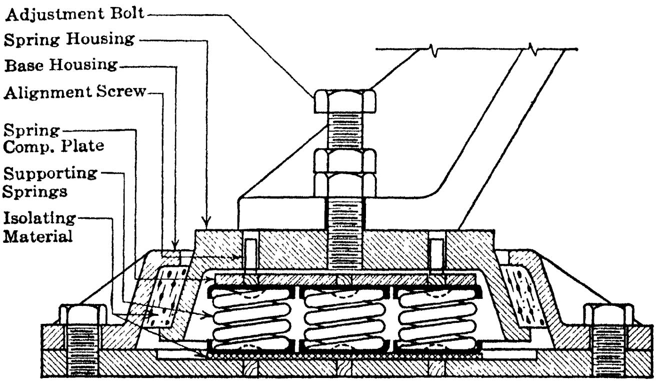


FIG. 5·5.

5·5. Commercial Types of Suspensions. The elastic support for any machine must be carefully designed according to definite rules. Experience, together with a knowledge of theory, is necessary to isolate a machine effectively. Not just any elastic material or quantity of it will do since an incorrectly designed suspension may actually magnify the force transmitted. Because the choice between various materials or types of isolation is often hard to make, some of the advantages and disadvantages of the more common ones are listed.

Metal Springs. A wider range of vibrations can be isolated by metal springs than by any other means known. This is due to the large variation of deflections obtainable by changing dimensions and material.

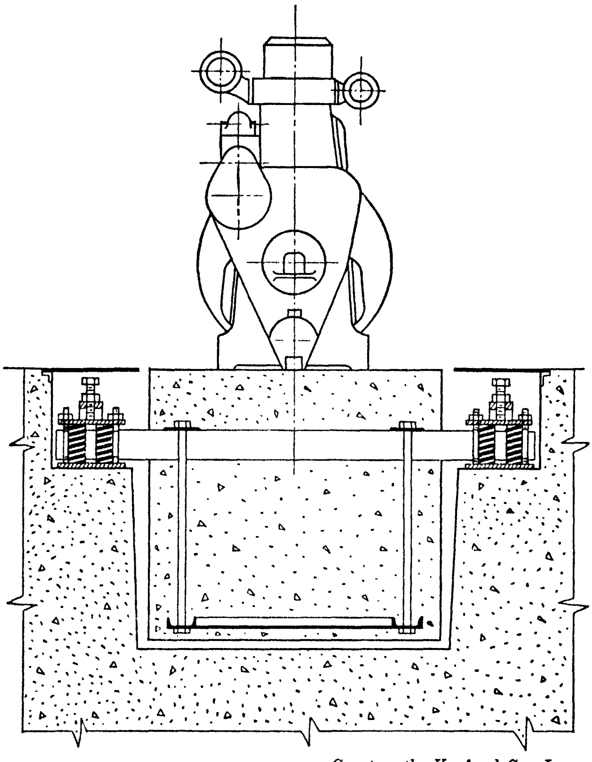


Courtesy the Korfund Co., Inc.

FIG. 5·6.

If one spring size does not work effectively it may usually be removed and another one put in. This is a great advantage. It has been found that springs have the best stability when the working height is about 1 to 1.5 times the outside diameter. For heavier loads several springs are grouped together into a unit (see Fig. 5·6) to give better stability.

Units of this type are available that will carry 16 tons. In this connection it might be mentioned that masses weighing up to 450 tons have been satisfactorily isolated with springs. Metal springs are very reliable and are not particularly affected by temperature, oil, or other conditions that would impair the continued operation of some other types. They have a very low damping coefficient so that little energy is lost. Because



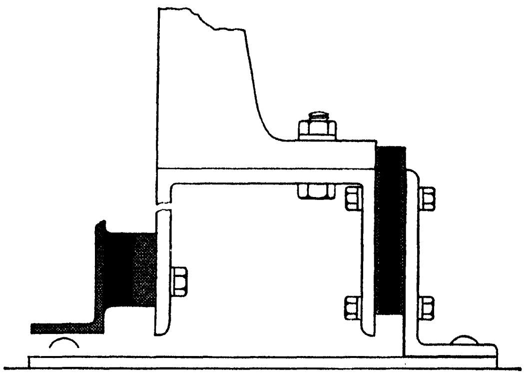
Courtesy the Korfund Co., Inc.

FIG. 5·7.

large amplitudes of motion may be encountered while passing through the resonant conditions it is necessary for some types of spring units to incorporate adjustable elements to give them a greater degree of damping. Often, too, some form of stops is provided to limit the motion.

One disadvantage of springs is that they transmit sound very well. To eliminate this difficulty, a layer of rubber or felt is often placed between the spring and the foundation, as shown in Fig. 5·6. All bolts, pipes, etc., connecting the machine and the foundation must be isolated to obtain the minimum noise and vibration. One typical application of springs for isolating a machine is shown in Fig. 5·7.

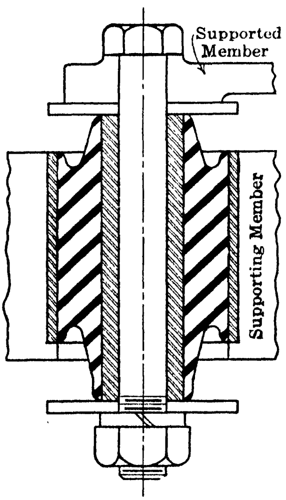
Rubber. Light machinery may be isolated very effectively with rubber. Rubber in compression carries considerably heavier loads than rubber in shear but with much lower deflections. Isolators with rubber



Courtesy the B. F. Goodrich Co.

FIG. 5·8.

in shear are particularly desirable because of the greater deflections possible. Two typical examples are the sandwich type shown in Fig. 5·8 and the tube form shown in Fig. 5·9. They give considerable deflection in the vertical direction but very little in the lateral because the rubber is then in compression or tension. Rubber in shear follows Hooke's law approximately if the loads are moderate but deviates considerably under compression.



Courtesy Lord Manufacturing Co.

FIG. 5·9.

Likewise the dynamic properties may vary considerably from the static properties. For instance, the stiffness of the harder compounds under dynamic load may approach 2 times the stiffness of the same rubber compound under a static load, but for the compounds more generally used it is below 1.25. The damping properties vary with the load, temperature, and frequency. Rubber serves very well to isolate the high frequencies of sound so that if no direct metal contact is made between the machine and the floor little high frequency vibration or noise is transmitted. Although the damping is not great enough to prevent excessive stresses or amplitudes, it is possible to incorporate

rubber stops to limit the motion without excessive shock. The loading influences the expected life of the rubber; heavy loads cause destruction much faster than light loads. High-grade rubber properly protected may be expected to last 5 years and frequently 10 years.

Because of the great variety of isolation available, little can be said about its application. Reference should be made to manufacturers' ratings when rubber isolators are used. Rubber cannot isolate low frequencies as well as metal springs. One interesting application is for isolating a torsional vibration, as shown in Fig. 5·10. The rubber allows torsional movement rather easily because it is in shear but restricts the lateral motion because it is in tension or compression. This particular isolator has been used in a large variety of places where power is transmitted by shafts as it eliminates torsional vibration.

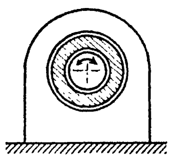
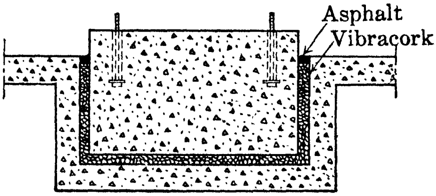


FIG. 5·10.

Natural rubber compounds should not be used above 150° F, and oil must be avoided. Synthetic rubber stands higher temperatures but tends to harden. It is not affected, however, by petroleum products.

Cork. One of the oldest materials used for commercial isolation is cork. It is generally used in compression, but occasionally it may be used in shear if loaded in compression at the same time. Natural cork blocks may be used for small isolators, but the large slabs are made from ground-up cork bonded together to give uniform structure and properties. By controlling the bonding process, several densities may be made to cover different applications, for the deflection varies inversely

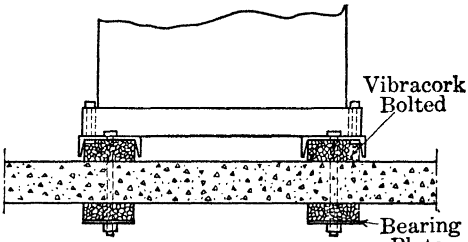


Courtesy Armstrong Cork Co.

FIG. 5·11.

as the density. The properties of cork change under dynamic load as do those of rubber. It is an excellent material to prevent transmission of sound but again through-bolts, pipes, etc., must be isolated or avoided for best results. It is used a great deal for insulation and acoustical purposes in buildings. In machine isolation it is valuable because its relatively high damping reduces vibrations at resonance conditions. It is nearly always necessary to mount large masses to get the required deflection

for good isolation. The machine is usually mounted on a large block of concrete, and then the whole block is separated from the rest of the building by a layer of cork slabs. These slabs vary from 1 to 6 in., but 2 to 4 in. are the thicknesses that are most common. Only the heaviest impact requires the thickest layers of cork. Oil, water, and

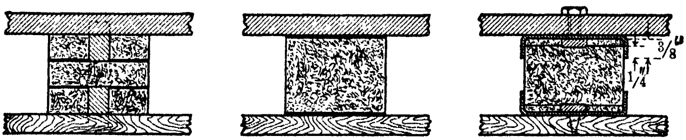


Courtesy Armstrong Cork Co.

FIG. 5·12.

moderate temperature have little effect upon the operating characteristics or useful life of cork. It does compress gradually with age, but, even so, cases are known where it has continued to operate for thirty or more years. Figures 5·11 and 5·12 show typical examples of cork isolation. Little definite data are available on the application of cork. What are available give rather inconsistent results. Low frequencies cannot be effectively isolated.

Felt. Large thicknesses of felt have good deflection properties. It has very good damping qualities and prevents sound transmission very effectively. These properties make felt suitable for isolating at resonant frequencies and for use with metal springs to reduce direct noise transmission. Felt is used in the form of mats for sound insulation or small



Courtesy American Felt Co., Glenville, Conn.

FIG. 5·13.

pads to put under machines. Usually the loading is limited to 1 to 50 lb/in.^{2}, but for small pads the pressure may rise to several hundred pounds per square inch. Figure 5·13 shows different methods of mounting felt to prevent sidewise motion.

When felt is used certain limits of pressure are desirable. It is

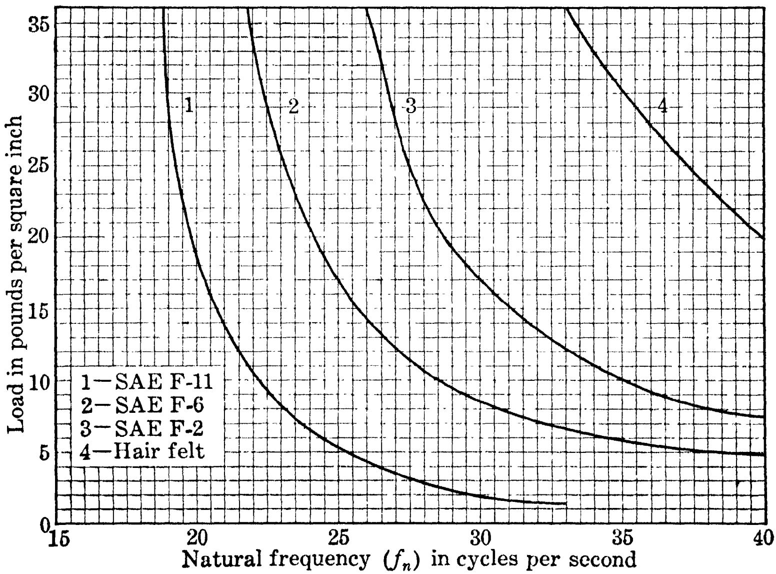
necessary, therefore, to express the natural frequency in terms of the pressure. The static deflection is given as

$$\delta_{st} = \frac{W}{A}\cdot\frac{t}{E} = \frac{pt}{E}$$

When this expression is substituted in equation 2·27, the frequency is given as

$$f_n = \frac{1}{2\pi}\sqrt{\frac{gE}{pt}} \qquad [5\cdot5]$$

The only difficulty involved in using this equation is that E varies with p. This may be overcome by plotting curves. The values of f_n for four



Courtesy American Felt Co., Glenville, Conn.

FIG. 5·14.

Recommended pressures for felt:

SAE F-11	1–6 lb/in.^{2}
SAE F-6	6–12 lb/in.^{2}
SAE F-2	12–25 lb/in.^{2}
Hair felt	over 25 lb/in.^{2}

grades of felt are indicated on Fig. 5·14. These curves are based upon felt 1 in. thick because that is the most commonly used thickness. Felt, like cork, cannot isolate low frequencies as well as rubber or springs.

Illustrative Problem. Considerable noise is produced by a machine weighing 46 lb and operating at a speed of 3600 rpm. What felt would give 80% isolation?

Solution. The natural frequency required can be determined from the transmissibility equation 5·3.

$$\text{T.R.} = \frac{1}{\left(\dfrac{\nu}{\omega_n}\right)^2 - 1}$$

$$0.20 = \frac{1}{\left(\dfrac{\frac{3600}{60}}{f_n}\right)^2 - 1}$$

$$f_n = 24.4 \text{ cycles/sec}$$

If a layer of SAE F-11 felt 1 in. thick is used, the load per square inch required is 6 lb/in.² The area of felt required is

$$A = \frac{W}{6} = \frac{46}{6} = 7.66 \text{ in.}^2$$

Thus, four pads, each having an area of 1.92 in.², could be used to support the machine. If a layer $1\frac{1}{2}$ in. thick were tried, the term used on the chart would be $f_n\sqrt{t}$ instead of f_n. This value is

$$f_n\sqrt{t} = 24.4\sqrt{1.5} = 29.9$$

This indicates that a pressure of 2 lb/in.² is required, giving a total area of 23 in.²

5·6. General Characteristics of Elastic Suspension. An isolated system has six degrees of freedom, each of which has its own natural frequency. Usually only the motion in the direction of the principal disturbing force is considered because the other frequencies are difficult to determine accurately.

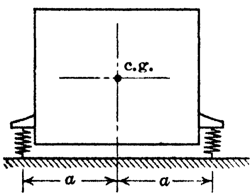


FIG. 5·15.

Several illustrations can be used to show some of the principles to follow for good isolation. First, it is desirable to have the center of gravity of the isolated structure midway between the supports whether mounted horizontally as shown in Fig. 5·15 or vertically as in Fig. 5·16. If this is impossible, the mounts must be selected so that each gives the same static deflection. The higher the center of gravity is above the mounts, the less stable the structure becomes. Often links must be added as shown in Fig. 5·17 to give the isolated body stability.

These same links may also be used to maintain fixed center distances between shafts. If the mounts can be placed in the plane of the center of gravity it is desirable, although not always practical or convenient. For heavy impact, such as is found in large hammers and forging machines, it is best to have the center of gravity below the plane of the isolators to give greater stability. This is quite easy when a large concrete block is necessary to give the required inertia mass. Many times it is necessary

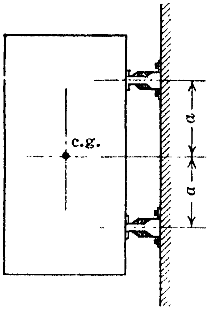


Fig. 5·16.

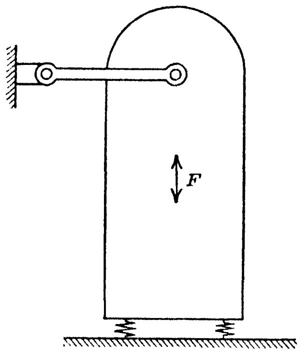


Fig. 5·17.

to mount two or three machines on the same base or substructure to maintain center distances for gears or belts. The whole assembly can then be isolated.

Actual experience shows that the calculations made are not always very accurate so that experience and judgment, together with a knowledge of theory, are necessary. Damping coefficients and their effects are particularly hard to evaluate. Metal springs with their low value of damping are perhaps the easiest to calculate. Behavior of other materials becomes more difficult to predict because of the varying damping and elastic properties. Because of the uncertainty of the calculations, it is often desirable to be able to change the characteristics of the isolating system. In isolating delicate instruments it is necessary to have very low natural frequencies (in the order of a cycle per second) to eliminate the low harmonics transmitted by buildings and the ground. Higher harmonics that cause noise must be eliminated by placing a layer of sound-insulating material between metal parts. The effect of pipes, conduits, and through-bolts must be eliminated as far as possible to prevent noise and force transmission.

Damping is very desirable near the resonant speed to reduce the amplitude of motion. Leaf springs have considerable damping due to friction between the leaves. Special devices are often added to intro-

duce damping to a system. Dash pots and automotive shock absorbers are forms of fluid damping devices. Internal or hysteresis damping is another form of considerable importance in engineering. Crankshafts are dependent upon internal damping to prevent excessive torsional vibrations. Crankshafts of cast iron were introduced because cast iron has much better damping properties than steel.

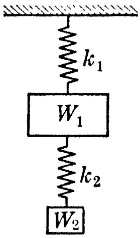


FIG. 5·18.

5·7. Undamped Dynamic Vibration Absorbers. If a body is vibrating, it is possible to reduce or eliminate this vibration by suspending another weight on an elastic member from the original system. The system is then fundamentally a two-mass system such as is shown in Fig. 5·18. The amplitude of motion of the body 1 is given in equation 4·27.

$$x_{01} = \frac{\dfrac{F}{k_1}\left[1 - \left(\dfrac{\nu}{\omega_{n2}}\right)^2\right]}{\left[1 - \left(\dfrac{\nu}{\omega_{n2}}\right)^2\right]\left[1 + \dfrac{k_2}{k_1} - \left(\dfrac{\nu}{\omega_{n1}}\right)^2\right] - \dfrac{k_2}{k_1}} \qquad [5\cdot6]$$

where $\omega_1 = \sqrt{\dfrac{k_1 g}{W_1}} \qquad \omega_2 = \sqrt{\dfrac{k_2 g}{W_2}}.$

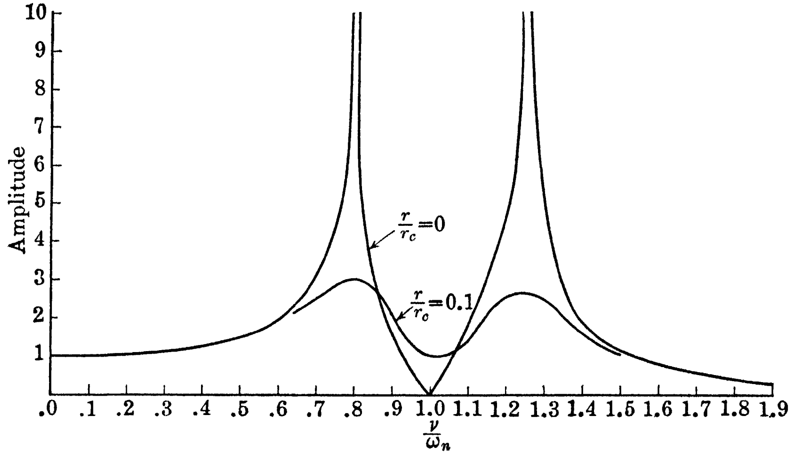


FIG. 5·19.

The plot of equation 5·6 is shown in Fig. 5·19. Obviously the amplitude x_{01} will be zero when $\left[1 - \left(\dfrac{\nu}{\omega_{n2}}\right)^2\right] = 0$. This means that the amplitude $x_{01} = 0$ when $\nu = \omega_{n2}$. Thus, if W_1 is acted upon by a

frequency $\nu = \omega_{n1}$, the amplitude of W_1 may be reduced to zero by adding a second spring and weight such that $\omega_{n2} = \omega_{n1}$. This may be done by making $\frac{k_2}{W_2} = \frac{k_1}{W_1}$. The amplitude of the second weight under the same conditions, which can be found from equation 4·28, is

$$x_{02} = -\frac{F}{k_2}\sin \nu t$$

This shows that the force exerted by the second spring on W_1 is equal to and opposite the disturbing force. Figure 5·19 shows that the amplitude of W_1 is nearly zero over a very limited range of speeds. Therefore, absorbers of this type are used on systems where the disturbing frequency is very near the natural frequency of the original single-mass system and when the machine will operate over a very limited speed range.

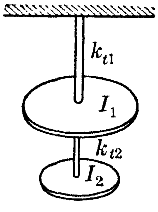


FIG. 5·20.

The same idea may be applied to torsional systems as shown in Fig. 5·20. In this case $\frac{k_{t2}}{I_2}$ must equal $\frac{k_{t1}}{I_1}$. One form of vibration absorber often encountered is shown in Fig. 5·21. This too operates only near the resonant speed and gives two other resonant speeds corresponding to those of the two-mass system indicated in Fig. 5·19. Because of its narrow range of effectiveness it has been generally replaced by a pendulum type absorber. Schematically the pendulum absorber may be

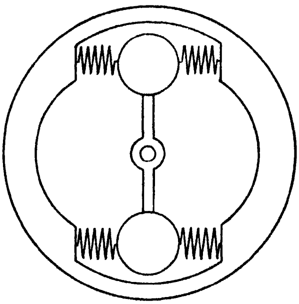


FIG. 5·21.

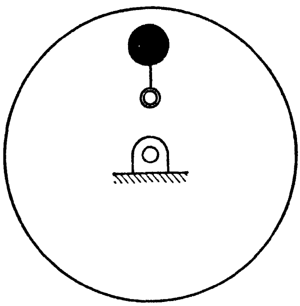


FIG. 5·22.

represented by Fig. 5·22, where the pendulum weight W is pivoted about a point A. During rotation the weight is acted upon by a centrifugal force equal to $\frac{W}{g} r\nu^2$. Therefore, in this type of absorber the acceleration $r\nu^2$ replaces the acceleration due to gravity of the ordinary

pendulum. The resonant frequency for the ordinary pendulum acted upon by gravity is given by equation 2·32.

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{R}} \qquad [5\cdot 7]$$

The resonant frequency for the pendulum absorber is, therefore, proportional to $\nu\sqrt{\frac{r}{R}}$. The pendulum can then be proportioned so that it is tuned to act as an absorber at any speed. Because the length of the

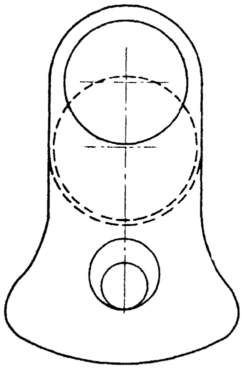


Fig. 5·23.

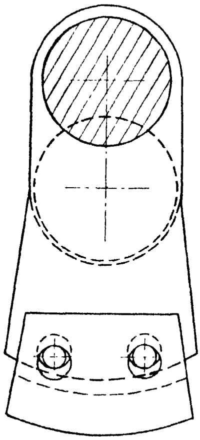


Fig. 5·24.

pendulum needed is not practical, several ingenious equivalent mechanisms have been substituted. Two are shown in Fig. 5·23 and Fig. 5·24.

Illustrative Problem. A motor weighing 10 lb and running 1800 rpm is mounted at the end of a cantilever bar. The bar is 4 in. wide and $\frac{15}{16}$ in. deep and has an effective length of 30 in. As the amplitude is excessive it is proposed to add another cantilever to act as an absorber. What proportions should it have?

Solution. From equation 2·35 the spring constant for the main cantilever is

$$k = \frac{3EI}{L^3} = \frac{3(30)10^6(\frac{1}{12})4(\frac{15}{16})^3}{30^3} = 915 \text{ lb/in.}$$

Any weight may now be selected to serve as the second weight. Let us assume that we will use a $\frac{1}{2}$-lb weight. The value of k for the second cantilever may now be determined from the

$$\frac{k_1}{W_1} = \frac{k_2}{W_2}$$

or

$$k_2 = \frac{k_1 W_2}{W_1} = \frac{915(0.5)}{10} = 45.7 \text{ lb/in.}$$

Thus a bar with a spring constant equal to 45.7 lb/in. can be used to mount the $\frac{1}{2}$-lb weight on the first mass. It is well to make the system adjustable so that it can be tuned to act most effectively. Approximations and small errors in measurements may be compensated for in this manner.

5·8. Damped Vibration Absorbers. The amplitude at any resonant speed is reduced considerably by damping. This is shown by the curve in Fig. 5·19. The mathematical theory of damped vibration absorbers is very complicated and will not be considered here. The problem of obtaining the correct amount of damping is difficult from a practical viewpoint. Many automotive crankshaft torsional dampers operate on this principle. The energy dissipated as heat through the damping of the absorbers would otherwise be used to vibrate the shaft.

5·9. Removing the Cause of the Vibration. Many times the cause of a vibration may be removed by careful study and analysis. Rotational unbalances may be reduced, but it is impossible to balance any machine completely. Static balance is not sufficient; only under dynamic conditions may the unbalance be reduced to an acceptable degree. Various commercial balancing machines are available that indicate the unbalance automatically.[20,21] When a shaft is not uniformly stiff for all angles of rotation a periodic motion is produced that can be removed only by giving it a uniform stiffness.[22] Magnetic forces in electric machinery often cause deflections which may grow to large proportions at resonance.[23] Propellers on airplanes or boats are very sensitive to variations of pitch between the blades.[24] Any number of cases may be cited to illustrate the presence of different causes for vibration, but usually the cause may be found and reduced. The amount of unbalance to be tolerated and the amount of isolation to be used depend upon the relative cost, for often it is cheaper to allow some unbalance and isolate the machine from its support rather than to eliminate the unbalance and avoid the necessity of isolation.

5·10. Vibration Instruments. Vibration instruments can be classified as those used for frequency measurement and those used for motion analysis. In many cases only the frequency is desired. For very slow vibrations counting by observation or by mechanical and electrical counters may suffice. Tachometers may be used for determining rotational speeds. Reed tachometers employing cantilever reeds graduated to respond to different impressed frequencies indicate frequencies very closely. Stroboscopes in which short flashes of light are synchronized with the vibration so that the vibrating or rotating body appears to

stand still are also commonly used to measure frequency. Variations of this are the mechanical instruments where a slotted disk is spring-driven at very closely governed speeds. When the moving body is viewed through the slots at the synchronous speed the eye sees it every time in the same position, and it appears to stand still. Other instruments discussed in succeeding paragraphs may also be used to indicate frequencies.

It may be desirable to measure many different quantities in addition to frequencies. Among these quantities are displacements, velocities, acceleration, wave form, and noise level. Most quantities must be magnified by mechanical, electrical, optical, or a combination means.

Very large vibrations may be measured by observation with or without a measuring microscope. Large amplitudes may be recorded by allowing a pencil attached to the vibrating member to mark on a piece of paper that is held fixed, or vice versa. Most vibrations are too small for these methods, and so mechanical vibrometers such as mentioned in Chapter III may be used. The relative motion of the mass to the frame is recorded by dial gages in the simpler types or by recording pens actuated by levers that magnify the motion in the more complex types. A paper strip is driven at a uniform speed under the pen. Timing waves are put on the same paper strip for comparing frequencies.

Electric instruments can also be used for measuring displacements. Electric pickup units are used to produce a change in some electrical quantity. This change is magnified and indicated or recorded by an oscillograph or meter. Among the electric pickups that indicate displacement are those employing electromagnetic and resistance changes. The electromagnetic type depends upon the change in flux produced by a change in air gap, whereas the resistance type depends on a change in resistance when carbon, a wire, or other material is deformed. Many times stress is more important than displacement in which instance wire strain gages are proving very popular.

Electric pickups which indicate velocity usually depend upon moving a coil through a magnetic field so that the voltage generated will depend on the rate at which the lines of flux are cut. Acceleration is usually indicated by a piezoelectric crystal which produces a potential across two surfaces when it is bent. Bending of the crystal is often effected by the inertia force from a small weight placed on one corner. Microphones together with their amplifiers are used for indicating noise levels.

Optical instruments usually depend on a mirror. When a light is thrown on the mirror, the reflection indicates the motion greatly magnified on a ground glass or photographic paper. In some special types of

optical instruments a photocell is employed to measure light intensity.

Special auxiliary equipment is very often used with the above equipment. Differentiating or integrating circuits can be used to change displacement, velocity, or acceleration to one of the other quantities.

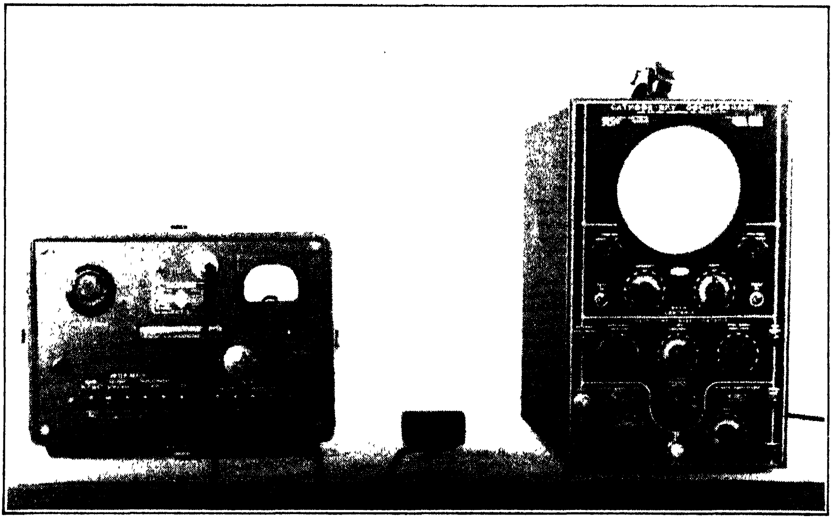


FIG. 5·25.

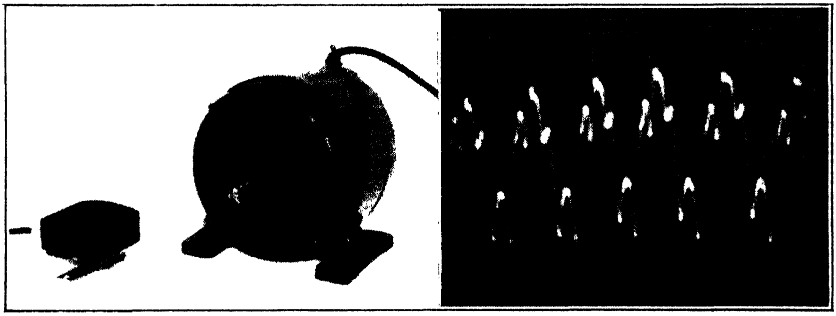


FIG. 5·26.

Analyzers can be used to study the wave form. These depend either on filtering undesired frequencies out or tuning a circuit to pick out the desired frequency. This analysis indicates the relative magnitude and importance of the different frequencies and thereby locates sources of trouble.

The equipment shown in Fig. 5·25 consisting of a vibration meter (left), a crystal type pickup (center), and an oscillograph (right) can be

used for both measuring and getting a picture of a vibration. Figure 5·26 shows a motor mounted on a board together with the pickup. The corresponding record as photographed on the oscillograph is shown on the right-hand side of this figure. This record shows that the vibration resulting from running of the motor consists of a large low frequency vibration on which is superimposed a high frequency vibration. Figure 5·27 shows the corresponding vibration record when the motor is mounted on a cork pad. In this case the high frequency vibration is eliminated but the low frequency component is transmitted. Figure 5·28 shows the motor mounted on commercial rubber isolators. In this case the low frequency component has been absorbed, but a portion of the high frequency component has been transmitted.

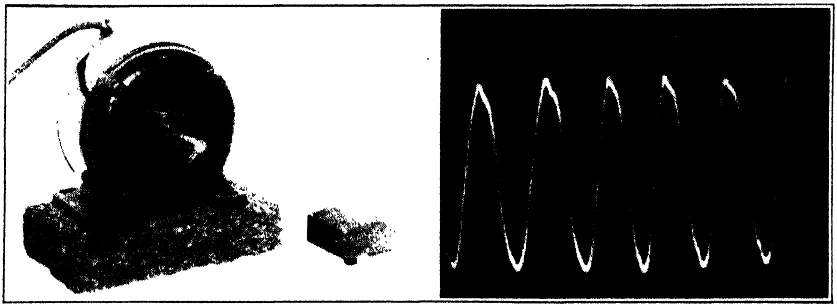


FIG. 5·27.

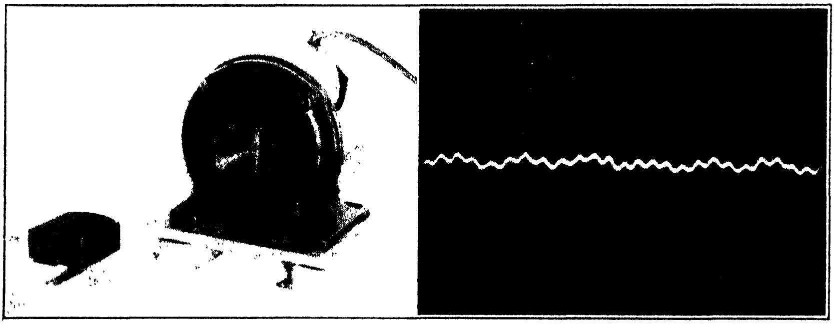


FIG. 5·28.

PROBLEMS

5·1. A refrigerating unit weighs 82 lb. The reciprocating compressor operates at 575 rpm. What spring constant k should each of four supporting springs have in

order that only 10% of the shaking force should be transmitted to the supporting structure? If the motor operates at 1725 rpm, what amount of motor unbalance force is transmitted to the frame by these same springs?

5·2. An engine operates at such a speed that the main disturbing frequency is 1425 cycles/min. The engine and its base weigh about 11,400 lb and are supported equally by 6 springs, each of which has a spring constant equal to 20,000 lb/in. What is the transmissibility? What is the percentage of isolation? What spring constant and static deflection are necessary to give 90% isolation?

5·3. A motor-pump unit operates at 1725 rpm. It seems desirable to allow only about 15% of the unbalance force to reach the floor. The unit weighs 860 lb and will be supported by 4 isolators. What load rate should the isolators have for these conditions? This corresponds to what ratio of forced frequency to natural frequency?

5·4. A measuring instrument is to be isolated from a vibration owing to a motor near by running at 960 rpm. The unit weighs 35 lb. What static spring deflection is necessary to limit the transmissibility to 20%? 10%? 5%?

5·5. A radio unit weighing 24 lb is mounted on 4 rubber isolators each carrying the same load. Near by is a rather heavily unbalanced motor running at 1760 rpm. The designed load for each isolator now used is 6 lb with a deflection of $\frac{1}{16}$ in. Because it seems desirable to reduce the transmissibility it is proposed to use a set of rubber mounts designed for a 6-lb load and a $\frac{3}{32}$-in. deflection. What would be the change in transmissibility?

5·6. A large slow-speed compressor and its base weigh 45 tons. It runs at 360 rpm. What deflection is necessary to give a natural frequency such that the frequency ratio will be 3? What is the percentage of isolation? What type of isolator would have to be used?

5·7. A small instrument weighing 28 lb must be isolated from the supporting table which vibrates at a frequency of 1720 cycles/min. Therefore, it is mounted on 4 rubber isolators that deflect $\frac{1}{8}$ in. The ratio of damping constants for rubber is $r/r_c = 0.05$. Compare the transmissibility when damping is considered with the transmissibility neglecting damping.

5·8. A large Diesel engine in a ship is mounted on a base which is supported equally on 8 spring isolators in order to prevent the vibration from being transmitted throughout the ship. The combined weight of the engine and its base is 16 tons. The operating speed is always above 100 rpm. What spring constant must each isolator have to give 80% isolation? What transmissibility is obtained by having a frequency ratio $\nu/\omega_n = 2.5$? 3? 5?

5·9. Design a felt mounting for a textile machine weighing 520 lb and operating at a speed where the disturbing frequency is 4000 cycles/min. Since the main objection is the higher frequency noise, the transmissibility may be 35%.

5·10. Some of the modern passenger trains are supported by helical springs that give a pendulum suspension. These springs give a static deflection of about 10 in. If it is assumed that the train travels over tracks that are slightly wavy with successive crests 40 ft apart, what will be the undamped transmissibility if the train travels at a rate of 20 mph? 80 mph?

5·11. Since propellers for airplanes have such a large moment of inertia relative to the engine moment of inertia, they may be approximated by a single disk on a shaft. As the engine would operate at the resonant speed of the system, a vibration absorber of the form shown in Fig. 5·21 is added. This gives the equivalent system shown in

Fig. P5·11, where I_e = 12.2 lb-in.-sec² and k_{te} = 700,000 lb-in./radian. If the absorber shaft has a spring constant k_{ta} = 63,000 lb-in./radian, what is the required moment of inertia of the absorber?

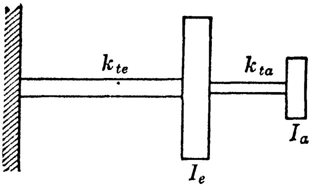


FIG. P5·11.

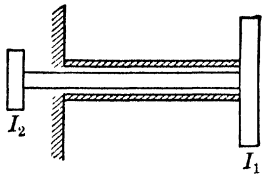


FIG. P5·12.

5·12. A solid disk I_1 weighing 26 lb and 10 in. in diameter is carried by a hollow steel shaft with a $1\frac{3}{4}$ in. outside diameter, a $1\frac{1}{4}$ in. inside diameter, and a length of 18 in. A second disk weighing 3 lb and 4 in. in diameter is carried on the end of a solid shaft extending through the hollow shaft as shown in Fig. P5·12. What diameter shaft 23 in. long is needed for I_2 to act as an absorber?

5·13. In problem 2·14 a 9-cylinder radial engine was mounted on 6 isolators to reduce the effects of torque impulses. What would the transmissibility be if the engine runs at 2200 rpm?

CHAPTER VI

EQUIVALENT SYSTEMS

6·1. Introduction. The calculation of critical speeds and relative motions for mechanical equipment becomes quite involved because many different types of mechanical elements are usually employed. Such systems do not readily lend themselves to analytical solution. It is possible, however, to reduce a general machine such as an airplane engine and transmission, a Diesel engine drive on a boat, or any other mechanical system or unitized machine to an equivalent disk and shaft system for the purpose of making a vibration analysis.[25] When reduced to such a simplified system the methods which have been developed or which are given in this chapter can be used to solve for critical speeds or relative amplitudes of motion.

The natural vibration of elastic systems involves the interchange of potential and kinetic energy. Any system must have elasticity and weight or inertia to be able to vibrate. The potential energy in the system is present as elastic energy stored in the shafts, belts, or other elastic members. The kinetic energy is stored in the weight or mass of the system by reason of its velocity. In a simple vibrating system consisting of a disk and a shaft, the disk is the inertia mass of the system where kinetic energy may be stored, and the shaft is the elastic member where potential energy may be stored. The solutions of the more complicated types of vibration problems to be considered here are based on the principle of replacing a complicated system by an equivalent system of disks and shafts which will have the same capacity for storage of kinetic and potential energy.

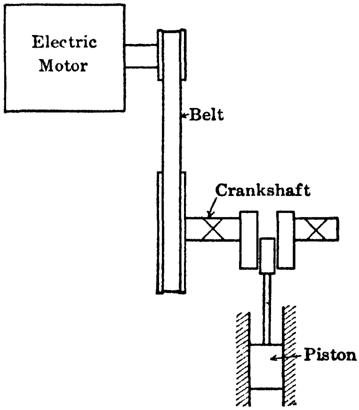


FIG. 6·1.

In a complicated system such as the compressor and drive shown in Fig. 6·1, the various parts operate at different speeds and have different elastic properties and weights. In a system such as this all parts are reduced to an equivalent rotating system.

The weights and elastic members are varied to give an equivalent energy system which in this case will be a disk and shaft system. In this equivalent system, all elastic members are reduced to an equivalent shaft of unit diameter and of such a length as to give an equivalent energy-storage capacity at the speed of the equivalent mass. The inertia mass is reduced to give an equivalent energy-storage capacity at the speed of the equivalent inertia disk. In cases of members having a variable inertia effect the equivalent inertia disk is chosen to give the average effect over a cycle of the variable inertia.

6·2. Equivalent Weight and Inertia Systems. The various rotating masses in the system shown in Fig. 6·1 operate at different speeds, and some of the masses, such as the engine piston, have motion of translation or have an oscillating motion instead of fixed rotation so that some method of converting these into an equivalent rotating disk is necessary. The general principle used in reducing these various mass systems to an equivalent simple rotating system is to make the conversion on the basis of equivalent kinetic-energy storage capacity.

The moment of inertia of a solid disk of uniform thickness is

$$I = \frac{Wd^2}{8g} \qquad [6\cdot 1]$$

where d = diameter of disk in inches

W = weight of disk in pounds

For a flywheel or pulley it is necessary to express the moment of inertia in terms of an equivalent diameter. For a pulley, where the weight and inertia of the arms can be neglected, the approximate moment of inertia, if the weight is assumed to be concentrated in the rim, is

$$I \text{ (approx.)} = \frac{W_r d_r^2}{4g}$$

where d_r = mean diameter of the rim in inches

W_r = weight of the rim in pounds

Where more exact calculations of the moment of inertia are desired for a pulley or flywheel or for any other type of weight the results are normally given in terms of the individual Wr^2 or Wd^2 values. In this case we have

$$I = \frac{\Sigma Wd^2}{4g} = \frac{\Sigma Wr^2}{g} \qquad [6\cdot 2]$$

where d = mean diameter of individual weight particles
r = mean radius of individual weight particles
W = weight of particle concentrated at d or r

The Σ sign indicates the addition or summation of individual values.

The moment of inertia can frequently be obtained experimentally by the methods given in section 2·13. Where such methods can be applied they will give more accurate results, particularly for complicated shapes.

The kinetic energy stored in a rotating weight is proportional to the angular velocity squared. Then the equivalent inertia of weights having rotary motion is given by

$$I_e = \frac{(\text{Speed of actual weight})^2}{(\text{Speed of equivalent weight})^2} I \text{ (actual)} \qquad [6\cdot3]$$

By using this formula, it is possible to reduce the mass directly to the required speed. This general formula applies regardless of the method of driving the mass, that is, whether through gear, belt, linkage, or chain. The above formula can, therefore, be used to reduce any rotating mass to the equivalent system speed. The speed of the prime mover is generally the most convenient one to use for the equivalent system speed.

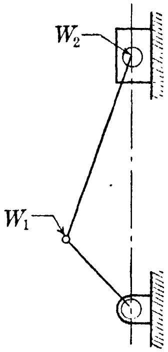


FIG. 6·2.

For reciprocating weights, it is necessary to resort to approximations inasmuch as the actual systems become very difficult to solve mathematically. The slider crank mechanism, as shown schematically in Fig. 6·2, is one typical case where it is necessary to reduce the mass of the connecting rod and the piston to an equivalent mass at the crank and then to any desired speed. The connecting rod weight can be approximated by placing part of its weight at the crankpin and the rest at the piston. The amounts to be placed at each point can be determined by taking moments about the center of gravity. If the dimensions shown in Fig. 6·3 are used the weights to be concentrated at the crankpin are

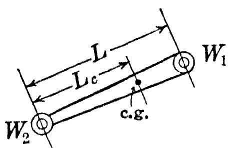


FIG. 6·3.

$$W_1 = \frac{W}{L} L_c \qquad [6\cdot4]$$

and that at the piston is

$$W_2 = \frac{W}{L} (L - L_c) \qquad [6\cdot5]$$

These values are only approximate. The equivalent dynamic system requires that one equivalent weight be placed at the center of percussion. Since this would not lead to a simple solution of the present problem it is necessary to use the arbitrary approximation indicated above. The weight at the piston, W_2, can be combined with the weight of the reciprocating parts, which may be the piston, crosshead, and piston rod. Since the piston assembly is at rest at each end of the stroke and reaches maximum speed only near the center of the stroke it can be assumed to be torsionally effective only one-half of the time. The equivalent weight of reciprocating parts at the crank is one-half the actual weight. The equivalent moment of inertia of the connecting rod and piston assembly at crank speed is

$$I_c = \left[\frac{W_1}{g} + \frac{W_2 + W_p}{2g}\right] r^2 \qquad [6\cdot6]$$

where W_p = weight of piston assembly
W_1 = weight of connecting rod placed at crankpin
W_2 = weight of connecting rod placed at piston
r = crank radius
I_c = equivalent moment of inertia of piston and rod

For most conditions where the ratio of crank to connecting rod length is small the formula is sufficiently accurate. Timoshenko gives a more accurate formula which takes into account the effect of the crank and connecting rod lengths.[26]

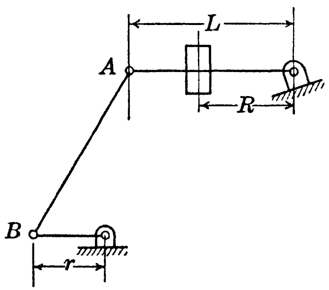


FIG. 6·4.

Figure 6·4 shows a typical 4-link mechanism where one crank rotates and the other crank member oscillates. A weight W on the oscillating member can be reduced to the crank by a method similar to that used for the slider crank mechanism. The first step in reducing such a system is to reduce the weight to the point A. Since the kinetic energy storing capacity varies as the velocity squared, the energy stored will vary as the radius of the weight squared. The equivalent weight at radius L will be obtained from the relation

$$W_c L^2 = W R^2$$

so that

$$W_c = W \frac{R^2}{L^2}$$

where W_c is the equivalent weight at a radius L. Since the weight only oscillates it is similar in effectiveness to the piston in the slider crank mechanism. The equivalent weight at the crank will then be reduced one-half, and the equivalent inertia at crank speed will be

$$I_c = \frac{1}{2}\frac{W_c}{g}r^2 = \frac{1}{2}\frac{W}{g}\left(\frac{R}{L}\right)^2 r^2 \qquad [6\cdot7]$$

6·3. Equivalent Elastic Systems. The members in a system, such as shown in Fig. 6·1, store elastic or potential energy. Since the various shafts rotate at different speeds and since some of the members, such as the belt, are non-rotating, it is necessary to reduce the system to an equivalent system of shafts rotating at the equivalent system speed. The equivalent shaft system includes both diameter and length, since they are both involved in determining the energy which can be stored. In order to simplify the problem, the shafts are all reduced to a unit diameter, which is a diameter of 1 in. The length is then determined to give the proper energy storage. The longer the equivalent shaft, the greater its energy-storing capacity.

If a unit diameter is used as a basis for calculation, it is necessary to vary the length, in order to maintain a constant k_t value or energy-storage capacity. This can be arrived at by referring to equation 2·34, from which

$$k_t = \frac{G}{L}\frac{\pi d^4}{32} = \frac{G}{L_e}\frac{\pi d_e{}^4}{32}$$

where G = torsional modulus of elasticity in pounds per square inch
L = shaft length in inches
L_e = equivalent shaft length
d = shaft diameter in inches
d_e = diameter of equivalent shaft

If d_e is made unit diameter, and L_e denotes the equivalent length, we shall have for a uniform shaft

$$\frac{G}{L}\frac{\pi d^4}{32} = \frac{G}{L_e}\frac{\pi(1)^4}{32}$$

$$L_e = \frac{L}{d^4} \qquad [6\cdot8]$$

For a shaft which varies in diameter between supports, it is necessary to find the equivalent lengths of the various sections and add them together to obtain the total equivalent length of a shaft of unit diameter.

In the case of the shafts running at different speeds, it is necessary to find the equivalent spring constant k_t at the equivalent speed. This can be determined from the equation

$$k_t \text{ (equivalent)} = \frac{(\text{speed of actual shaft})^2}{(\text{speed of equivalent shaft})^2} k_t \text{ (actual)} \qquad [6\cdot 9]$$

Since the length of shaft is inversely proportional to the spring constant we can write

$$L_e = \frac{L}{d^4} \frac{(\text{speed of equivalent shaft})^2}{(\text{speed of actual shaft})^2} \qquad [6\cdot 10]$$

$$= \frac{L}{d^4} (\text{speed ratio})^2$$

A simple crankshaft is shown in Fig. 6·5. It is quite difficult to arrive at an equivalent length of a shaft of unit diameter for a crankshaft. A simplified empirical formula, as given by Carter, appears to be satisfactory for normal conditions and is used here.[27] By using the dimensions, as given in Fig. 6·5, this formula for the equal length L_{ec} can be expressed in the following form if unit diameter is assumed for the equivalent shaft:

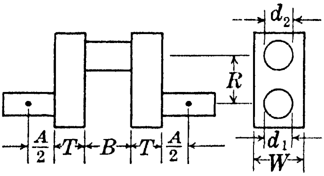


FIG. 6·5.

$$L_{ec} = \frac{A + 0.8T}{d_1^4} + \frac{0.75B}{d_2^4} + \frac{1.5R}{TW^3} \qquad [6\cdot 11]$$

A more exact formula has been worked out by Timoshenko,[27, 28] and other empirical formulas are given by Wilson.

Figure 6·6 shows an equivalent diagram for a belt or chain drive. In this case the belt tension P represents only the effective pull. It is assumed that the slack-side tension does not influence the problem since it supposedly puts a constant stress or energy into the system. The energy stored in the belt is given by

$$PE = \frac{1}{2} \frac{P^2 L}{A E_B}$$

and in the shaft

$$PE = \frac{T^2 L_e}{2GJ} = \frac{(PR)^2 L_e}{2GJ}$$

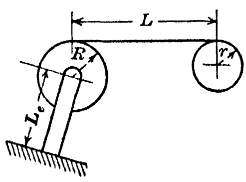


FIG. 6·6.

where A = area of belt
E_B = modulus of elasticity of belt
G = shear modulus of shaft
J = polar moment of inertia of shaft
R = radius of pulley
L = effective length of belt
L_e = equivalent shaft length at speed of pulley with radius R

Equating these and solving for L_e, we have

$$L_e = L\frac{GJ}{E_B A R^2}$$

For a shaft of unit diameter this reduces to

$$L_e = \frac{\pi L G}{32\, A E_B R^2} \qquad [6\cdot12]$$

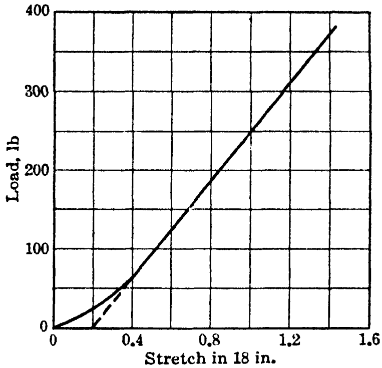


FIG. 6·7.

4-ply rubber belt; 1 in. × 0.23 in.; breaking load 1200 lb; permanent stretch at B.P. = ½ in. in 18 in.; E_B = 22,640 lb/in.^{2}; rupture stress = 5454 lb/in.^{2}

Figure 6·7 shows a plot of data obtained from a belt test. These data show that, after the initial stretch is taken up, the modulus of elasticity for the rubber belt E_B = 22,640 lb/in.^{2} This value will be used in determining the equivalent shaft length for a belt.

A beam of a 4-link mechanism, such as is shown in Fig. 6·8, can also be replaced by an equivalent shaft of length L_e and unit diameter. The potential energy in the shaft at the position shown, that is, when the crank r is parallel to the beam, is

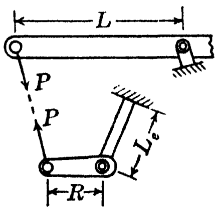


FIG. 6·8.

$$PE = \frac{1}{2}\frac{T^2 L_e}{GJ} = \frac{(PR)^2 L_e}{GJ}$$

where T = torque in the shaft
P = load in the pitman
J = polar moment of inertia of shaft

The energy stored in the beam is

$$PE = \frac{1}{6}\frac{(PL)^2 L}{EI}$$

If these are equal, we then have

$$\frac{1}{2}\frac{(PR)^2L_e}{GJ} = \frac{1}{6}\frac{(PL)^2L}{EI}$$

$$L_e = \frac{1}{3}\frac{L^3GJ}{R^2EI}$$

When the crank is vertical the torque in the shaft will be zero. The equivalent shaft length for that position would be zero so that the average value would be only one-half of the above value; hence

$$L_e = \frac{1}{6}\frac{L^3GJ}{R^2EI} \qquad [6\cdot 13]$$

Another method of justifying the equivalent shaft length and inertia of such a non-rotating member as a beam is to note that the natural frequency of a member is determined by the ratio of the elasticity to the mass or equivalent inertia. If, therefore, the effectiveness of one is reduced to half, the other should also be reduced to half.

Flexible couplings are frequently introduced in unitized systems to take care of misalignments. Because of their flexibility they can change the critical speeds of systems in which they are placed. If the spring constant of a coupling is k_{tc} and the spring constant of a shaft system in which it is used is k_{ts}, the equivalent spring constant, k_{te}, of the combination can be determined by noting that the angle turned through under torque T is

$$\frac{T}{k_{te}} = \frac{T}{k_{tc}} + \frac{T}{k_{ts}}$$

$$k_{te} = \frac{k_{tc}k_{ts}}{k_{tc} + k_{ts}} \qquad [6\cdot 14]^{*}$$

The equivalent length of shaft of unit diameter can be found from the relation for the spring constant of a shaft

$$k_{te} = \frac{G}{L_e}\frac{\pi(1)^4}{32}$$

$$L_e = \frac{G}{k_{te}}\frac{\pi}{32} \qquad [6\cdot 15]$$

In order to determine the effect of the coupling on the system it is then necessary to have a torque displacement curve from which k_{tc} can

be obtained. Some types of flexible couplings have considerable friction in them so that the damping they cause in the system helps to reduce the amplitude of the motion near the critical speed. A detailed analysis of couplings has been made by Ormondroyd and Wilson, and for further information these references should be consulted.[29,30]

6·4. Calculation of a Typical System. The critical speeds of a system depend upon the various masses and elastic members in the system. This makes it necessary to study each case separately. A typical system such as the compressor unit, Fig. 6·1, may be used to illustrate the method of solving for the equivalent systems and for the critical speeds. The data selected are given in Table 6·1.

TABLE 6·1

DATA FOR COMPRESSOR SYSTEM WITH A SUMMARY OF EQUIVALENT MOMENTS OF INERTIA AND ELASTICITIES

	Actual	Equivalent Values at Motor Speed
Motor speed, rpm	1800	
Motor pulley diameter, in.	10	
Compressor pulley diameter, in.	36	
Pulley center, in.	72	
Belt, 6-ply rubber (approx. 0.3 in. thick), 8 in. wide		
Armature shaft, armature to pulley, $1\frac{5}{8}$ in. diam., $10\frac{3}{4}$ in. long		
Connecting rod weight, lb	36	
Distance—C.G. of rod to crankpin, in.	6	
Length of rod, in.	18	
Weight of piston, piston rod, crosshead, lb	108	
Stroke, in.	9	
I-10-in. pulley, in.-sec^2	3.3	3.3
I-motor armature, lb-in.-sec^2	6.5	6.5
I-compressor pulley, lb-in.-sec^2	140	10.8
I-crank, piston, rod, lb-in.-sec^2	9.8	1.26
L_e equivalent lengths of unit diam.		
1. Armature to pulley, in.	1.55	1.55
2. Belt, in.	62.5	62.5
3. Pulley to crank, in.	0.15	1.94
k_t torsional rigidity		
1. Armature to pulley		7.6×10^5
2. Belt		0.1885×10^5
3. Pulley to crank		6.07×10^5

The following examples will illustrate the method of calculating the equivalent elasticities, masses, and finally the critical speeds.

The weight of the connecting rod must be divided according to equations 6·4 and 6·5 into two weights at the crankpin and wrist pin, respectively, so that

$$W_1 = \frac{12(36)}{18} = 24 \text{ lb}$$

$$W_2 = \frac{6(36)}{18} = 12 \text{ lb}$$

The equivalent moment of inertia of the rod and piston is determined by equation 6·6 and becomes

$$I_c = \left[\frac{24}{g} + \frac{(12 + 108)}{2g}\right](4\tfrac{1}{2})^2 = 4.40 \text{ lb-in.-sec}^2$$

The moment of inertia of the crank must be added to this quantity. If the moment of inertia of the main bearing journals is neglected, the remaining parts are the crankpin, crankwebs, and the counterbalance. The expression for the moment of inertia of the crankpin about the axis of rotation is

$$I_p = \frac{1}{2}\frac{W}{g}r^2 + \frac{W}{g}x^2 = \frac{\pi(4\frac{1}{2})^2}{4}(3\tfrac{3}{4})\frac{0.283}{g}\left[\frac{1}{2}\left(\frac{4\frac{1}{2}}{2}\right)^2 + (4\tfrac{1}{2})^2\right]$$

$$= 0.99 \text{ lb-in.-sec}^2$$

where x is the distance from center of gravity to center of rotation. The moment of inertia for the crankweb is

$$I = \frac{10(2)6(0.283)}{386}\left[\frac{10^2 + 6^2}{12} + 2.25^2\right] = 1.44 \text{ lb-in.-sec}^2 \text{ per web}$$

The moment of inertia of the two counterbalance weights on the crank is 1.6 lb-in.-sec^2. The total moment of inertia concentrated at the crank is then approximately equal to 9.8 lb-in.-sec^2.

The other mass elements in the system are the large pulley, the small pulley, and the armature of the motor. The values of the moments of inertia of these parts are presented in Table 6·1.

The elastic elements of the system are the motor shaft, the belt, and the crankshaft. The length of the motor shaft is $10\frac{3}{4}$ in. and the diameter

is $1\frac{5}{8}$ in. so that the equivalent length for a unit diameter shaft is, by equation 6·8,

$$L_e = \frac{10\frac{3}{4}}{(1\frac{5}{8})^4} = 1.55 \text{ in.}$$

The equivalent length of shaft for the belt for 6-ft centers can be found approximately from equation 6·12

$$L_e = \frac{\pi 72(12{,}000{,}000)}{32(0.3)8(22{,}640)5^2} = 62.5 \text{ in.}$$

The equivalent length of the crankshaft, Fig. 6·9, between points C and E is by equation 6·11.

$$L_{ce} = \frac{2(1\frac{3}{8}) + 0.8(2)}{(4\frac{3}{4})^4} + \frac{0.75(3\frac{3}{4})}{(4\frac{1}{2})^4} + \frac{1.5(4\frac{1}{2})}{(2)6^3} = 0.031 \text{ in.}$$

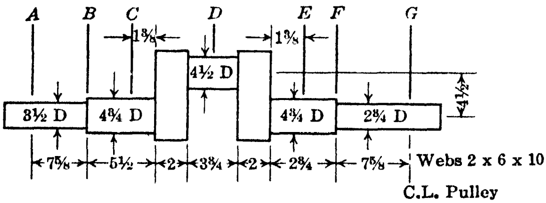


FIG. 6·9.

Only half of this is effective in our case, so $L_{de} = 0.015$ in. The equivalent lengths of shaft from E to F and F to G are given by equation 6·8.

$$L_{ef} = \frac{1\frac{3}{8}}{(4\frac{3}{4})^4} = 0.003 \text{ in.}$$

$$L_{fg} = \frac{7\frac{5}{8}}{(2\frac{3}{4})^4} = 0.133 \text{ in.}$$

The total equivalent length of shaft between the crank and large pulley is, therefore,

$$L_{dg} = 0.15 \text{ in.}$$

The difference in speed of the motor and the crank makes it necessary to correct the values of I or L. If we choose the motor speed as basic, the value of I for the large pulley is found by equation 6·3.

$$I \text{ (equivalent)} = \left(\frac{500}{1800}\right)^2 140 = 10.8 \text{ lb-in.-sec}^2$$

The value for the crank is indicated in Table 6·1. The belt elasticity was based on the small pulley so that it does not need to be corrected, but the equivalent length of the crankshaft must be corrected according to equation 6·9.

$$L_e = 0.15\left(\frac{1800}{500}\right)^2 = 1.94 \text{ in.}$$

These elements give a system, Fig. 6·10, which is equivalent from a vibration standpoint to the original system. This equivalent system may now be solved in the usual manner for disk and shaft problems to determine the critical speeds. Approximations are useful in solving for the frequencies. Because of the great elasticity of the belt the system is approximately a two-mass system. The lowest critical frequency is approximated by using the two-mass equation.

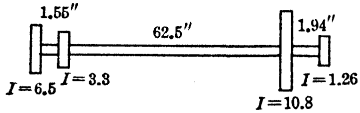


FIG. 6·10.

$$f = \frac{60}{2\pi}\sqrt{\frac{Gd^4(I_1 + I_2)\pi}{32I_1I_2L}}$$

$$= \frac{60}{2\pi}\sqrt{\frac{(12{,}000{,}000)1^4(12.06 + 9.8)\pi}{32(12.06)9.8(62.5)}} = 563 \text{ rpm}$$

This speed is close to the actual speed of the system, and the result might be a greatly overstressed member in the system. Two other critical speeds in this case could be approximated by considering the crank mass and the large pulley as one system and the armature and small pulley as the second system. This is reasonable since the large elasticity of the belt would almost eliminate the effect of one end of the system upon the vibration of the masses on the other end.

The values of these two critical speeds are

$$f = \frac{60}{2\pi}\sqrt{\frac{12{,}000{,}000(10.8 + 1.26)\pi}{32(10.8)1.26(1.94)}} = 7300 \text{ rpm}$$

$$f = \frac{60}{2\pi}\sqrt{\frac{12{,}000{,}000(6.5 + 3.3)\pi}{32(6.5)3.3(1.55)}} = 5620 \text{ rpm}$$

These values are in all probability accurate enough considering the accuracy of the original data. These approximations can be made only when it is certain the masses in question act as a two-mass system with a node between them.

If a system were considered as a three-mass system (that is, only three masses have much effect on the system as a whole), the equation for solving for the critical speeds is the same as equation 4·23.

$$\frac{I_1 I_2 I_3}{k_{t1} k_{t2}} \omega^4 - \left[\frac{I_1 I_2 + I_1 I_3}{k_{t1}} + \frac{I_2 I_3 + I_1 I_3}{k_{t2}}\right] \omega^2 + (I_1 + I_2 + I_3) = 0$$

where ω is in radians per second. The equation above may easily be solved as a quadratic for ω^2. A more accurate analysis could be made using the tabulation method.

Since the operating speed, 500 rpm, is close to the fundamental frequency of the system, some change should be made in the system. Many times the speed may be changed. Where the speed must be maintained, the mass or the elasticity of some element, or both, must be changed. In the example the belt would be the most logical element to change. Shortening the belt would raise the frequency and thus avoid the critical speed entirely. A heavier or lighter compressor pulley could also be used with similar results.

The example above was selected because it contained so many different elements. Equivalent systems are more commonly used for systems involving multicylinder engines geared or connected directly to generators, propellers, etc. A radial airplane engine having a supercharger and driving a propeller can be represented by a simple three-mass system wherein the supercharger is one mass, the pistons and connecting rod assembly of the radial engine form a second mass, and the propeller is the third mass. If the supercharger is geared up and the propeller is geared down it is necessary to convert the inertias and equivalent shafts to engine speed. An in-line engine used as a propeller drive would be represented by as many masses as there are cylinders plus the supercharger and propeller masses.

A marine installation having a six-cylinder engine, a flywheel, and a propeller would be represented by eight masses. A generator unit driven by an eight-cylinder engine with flywheel would have ten masses: eight for the engine, one for the flywheel, and one for the armature. If an exciter is attached to the end of the armature, another mass would be added. Systems of this type are simpler to reduce to an equivalent system than the example worked above in that fewer types of elements are involved. Because of the larger number of masses, they become more difficult to solve for critical speeds.

Figure 6·11*a* shows a more complex system involving a divided or branched shaft. Such systems are frequently encountered in such machinery as marine installations having dual propellers driven by a

common reducer or prime mover. These systems cannot be reduced to so simple an equivalent system as those indicated previously. They can, however, be reduced to such a system as is shown in Fig. 6·11*b*.

To solve for the natural frequencies of a branched system using the tabulation method, it is necessary to start with the mass at the end of one of the branches, such as mass I_1 in Fig. 6·11*b*. This mass is arbitrarily given an angular amplitude $\beta_1 = 1$ radian, just as in the simpler

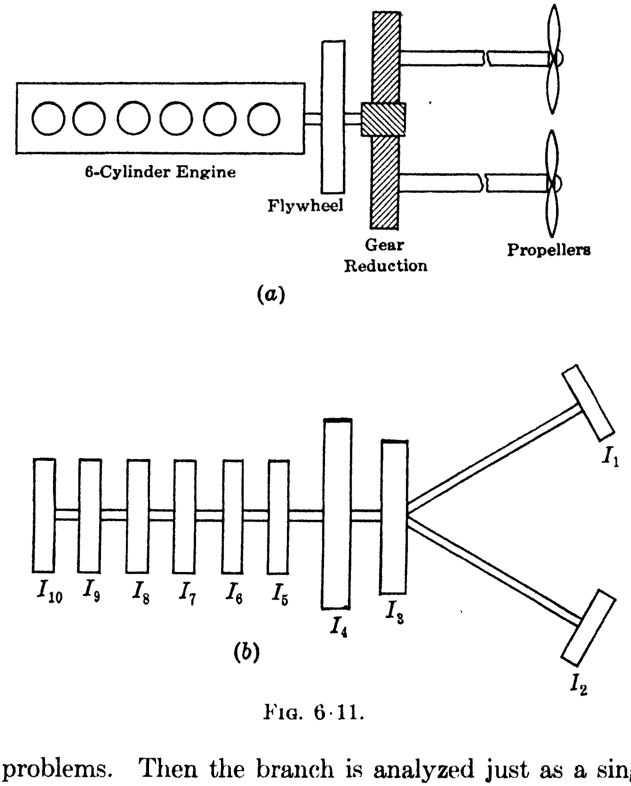


FIG. 6·11.

types of problems. Then the branch is analyzed just as a single shaft problem until the common disk I_3 is reached. Here it is found that I_3 has a definite amplitude β_3. If both branches are identical, β_2 will be equal to 1 radian in order that β_3 will be the same for both branches. If the branches are different it is necessary to assume temporarily that $\beta_2 = 1$ radian and then work to the junction and get a second value for β_3. It is impossible, however, for one mass to have two different displacements. We shall, therefore, make the second β_3 equal to the first. If we make them equal, all amplitudes on the second branch will be changed a proportional amount. We may proceed with these new amplitudes and determine the moment that each branch exerts on the common

single mass. After these two values are added, the remainder of the solution is the same as for a single-shaft problem. A different and often more convenient method of solution for such problems using the mobility method is given in Chapter VII.

PROBLEMS

6·1. A geared drive is shown schematically in Fig. P6·1. Set up the equivalent system at the speed of disk I_1 and make a sketch of it with the various lengths of

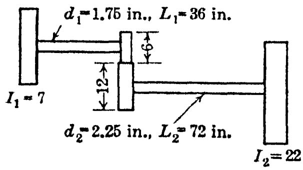


FIG. P6·1.

shafts and moments of inertia indicated on it. Neglect the mass of the gears. What is the natural frequency for torsional vibration?

6·2. In Fig. P6·2 the values corresponding to each element are

$I_1 = 15$ lb in. sec^2	$L_a = 15$ in.	$d_a = 2\frac{1}{2}$ in.
$I_2 = 4$ lb in. sec^2	$L_b = 10$ in.	$d_b = 1\frac{1}{2}$ in.
$I_3 = 5$ lb in. sec^2	$L_c = 38$ in.	$d_c = 2$ in.
$I_4 = 6$ lb in. sec^2		
$I_5 = 27$ lb in. sec^2		

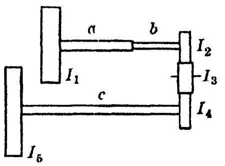


FIG. P6·2.

The speed of mass 2 is 240 rpm, the speed of mass 3 is 200 rpm, and the speed of mass 4 is 180 rpm.

(*a*) Sketch the equivalent system.
(*b*) Set up and determine the values of the equivalent moments of inertia at the speed of mass 2.
(*c*) Set up and determine the values of the equivalent lengths of shaft at the speed of mass 2.
(*d*) What are the resonant frequencies?

6·3. A radial airplane engine and propeller system reduces to an equivalent system of two masses. The moment of inertia, I_p, of the propeller is 150 lb-in.-sec^2 and that of the engine rotating parts I_e is 6 lb-in.-sec^2. The torsional stiffness of the shaft between the masses is 2,000,000 lb-in./radian. What is the natural torsional frequency? If I_p had been 200, what would the frequency have been? What if I_e had been 5? What would it be if the stiffness had been 3,000,000? What is the most

effective way of changing the natural frequency? If I_p is considered infinite, what is the natural frequency?

6·4. In the two-disk problem 4·5 the natural frequency is very near the operating speed so that a pendulum type absorber, such as is shown in Fig. 5·21, is attached to the second disk. This does not change the value of I_2 appreciably. The moment of inertia of the swinging weight is 1.9 lb-in.-sec^2 about the axis of the shaft. Each of the 4 springs has a spring constant k = 800 lb/in., and they act at an effective radius r = 5 in. Sketch the equivalent system and include the equivalent I's and k_t's on it. What are the natural frequencies of this system?

6·5. (*a*) Set up the equivalent system for the automobile drive shown in Fig. P6·5.

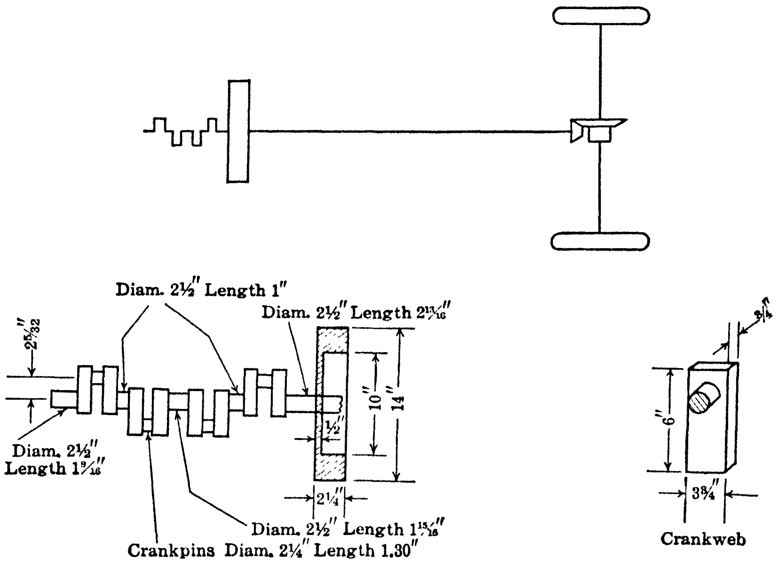


FIG. P6·5.

(*b*) Determine the two lowest natural frequencies. Why is it possible tc analyze the engine-flywheel system separately? The necessary information is:

Car weight, 3000 lb.

Piston weight, 1 lb 12 oz.

Connecting rod weight, 2 lb; length, 12 in.; center of gravity, $3\frac{5}{8}$ in. from crankpin.

Crank dimensions as shown.

For the drive shaft assume direct drive and that the shaft is 76 in. long and $1\frac{3}{4}$ in. in diameter from flywheel to differential pinion.

Differential gear ratio is 4.1 to 1.

Each side of the rear axle is 24 in. long and is $1\frac{3}{8}$ in. in diameter.

Wheel and tire, diameter 28 in., weight 40 lb. I is determined by suspending the wheel on a knife edge 7.69 in. from wheel axis and counting 42 complete cycles per minute.

6·6. Analyze the engine and flywheel system in problem 6·5 to find the lowest natural frequency.

6·7. Figure P6·7 shows a schematic drawing of a 7-cylinder radial engine drive. The dimensions are indicated on the drawing. The elasticity and mass of the gear reduction unit may be neglected. Additional data are:

Propeller: $I_p = 125$ lb-in.-sec^2.
Piston: Complete weight $W_p = 4.9$ lb.
Master rod: Weight $W_m = 14.5$ lb.
Length (center to center) $L_m = 10$ in.
C. of G. is 1.25 in. from crankpin.
Articulated rods: Weight $W_a = 2.5$ lb.
Length (center to center) $L_a = 7.5$ in.
C. of G. 3.6 in. from piston pin.
Gear reduction: Motor speed to propeller speed 3 : 2.

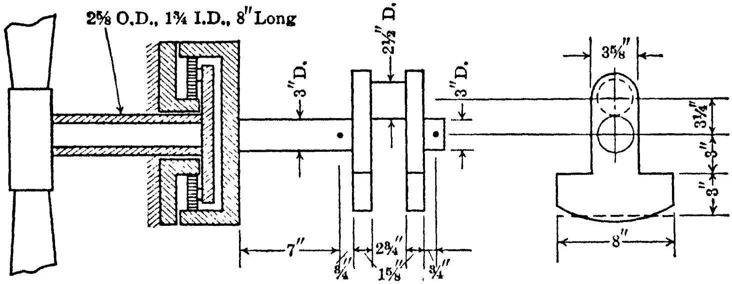


Fig. P6·7.

(*a*) Set up the equivalent system for this drive.
(*b*) Determine the lowest critical speed.

6·8. A small boat is driven by a single propeller drive. The drive is shown in Fig. P6·8. The inertia and elasticity of the gears are considered negligible. The

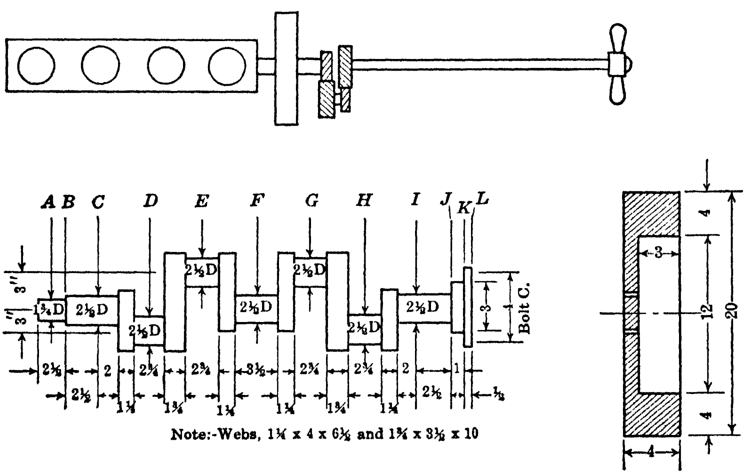


Fig. P6·8.

engine crankshaft and flywheel have the dimensions indicated. Additional data are:

Piston: Weight $W_p = 5.5$ lb.
Connecting rod: Weight $W_c = 7\frac{1}{2}$ lb.
Length (center to center) $L_r = 12$ in.
C. of G. is 9 in. from wristpin.
Propeller: $I_p = 0.8$ lb-in.-sec^2, including the effect of entrained water.
Propeller shaft: Length $L_s = 8$ ft 3 in.
Diameter $D_s = 1\frac{7}{16}$ in.
Drive shaft from flywheel to gear: Length $L_d = 9$ in.
Diameter $D_d = 1\frac{7}{16}$ in.
Gear reduction: 2 : 1.

(*a*) Set up the equivalent system for this drive.
(*b*) Solve for the two lowest natural frequencies.

6·9. Set up the equivalent system for problem 6·8 when a dual propeller drive is used instead of the single propeller drive. The same drive shaft and propeller as in problem 6·8 is used for both branches. Find the natural frequencies for this system.

CHAPTER VII

BEAMS

7·1. Introduction. A common type of vibration encountered in structures and machines is the lateral vibration of members acting as beams. Since such members are so common in structures and in machines (including shafting), methods of determining the frequency of such vibrations are included.

Beam vibration problems can be divided into several classes. The simplest of these is that of a beam carrying a single mass where the mass is sufficiently large that the weight of the beam itself can be neglected. The methods of solving such problems are discussed in section 2·15. These systems behave essentially as systems of a single degree of freedom. The second group includes problems where several masses are present and where the weight of the beam again is negligible. In this case relatively simple solutions can be obtained for the natural frequency using both graphical and analytical methods. The third class includes problems where the beam mass itself is relatively large or where the beam is loaded with a distributed mass. In this case a more general approach to the problem is necessary. This chapter will consider the general differential equations which apply to beam vibrations together with simplified methods of approximating the vibrations of such systems. There are a number of methods available for solution of more complex problems or of obtaining more exact solutions such as those of Stadola,[31] Myklestad,[32] and Proll.[33] These, however, are beyond the scope of this book and will not be discussed here.

7·2. Concentrated Weight on Beam. The analysis of a weightless simple beam carrying a concentrated weight was discussed in section 2·15. The solution for the natural frequency of weightless beams carrying a concentrated weight can be written in the general form

$$f = \frac{1}{2\pi}\sqrt{\frac{QEIg}{WL^3}} = \frac{3.13}{\sqrt{\delta_{st}}} \text{ cycles/sec} \qquad [7\cdot 1]$$

where f = natural frequency, in cycles per second

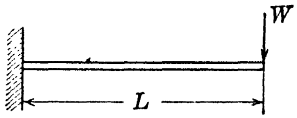
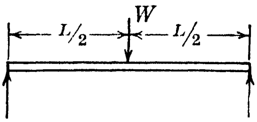
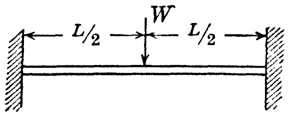
Q = beam constant given in Table 7·1

E = modulus of elasticity of beam material in pounds per square inch

I = sectional moment of inertia in inches^4
W = concentrated weight in pounds
L = beam length in inches
δ_{st} = static deflection in inches

This formula gives reasonably accurate results even when the weight of the beam is equal to that of the concentrated weight. A more accurate solution can be obtained by considering part of the beam weight as concentrated at the load. The amount depends upon the type of beam and the end conditions and will be discussed later.

TABLE 7·1

	Deflection	Q
Cantilever 	$\frac{WL^3}{8EI}$	3
Simply Supported 	$\frac{WL^3}{48EI}$	48
Fixed Ends 	$\frac{WL^3}{192EI}$	192

Illustrative Problem. Find the natural frequency of an aluminum alloy cantilever beam weighing 12 lb and having a moment of inertia of 2 in.^{4} The beam is 24 in. long and carries a concentrated load of 16 lb at its outer end.

Solution. If the beam is considered as weightless, the natural frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{3EIg}{WL^3}} = \frac{1}{2\pi}\sqrt{\frac{3(10{,}500{,}000)2(386)}{16(24)^3}}$$
$$= 53 \text{ cycles/sec or } 3170 \text{ cycles/min}$$

If the beam weight is taken into consideration the correct value is 2940 cycles per min. This is an error of about 7.5%, which may not be considered excessive under conditions in which the weights, dimensions, and physical properties are not accurately known.

7·3. Rayleigh Method for Beams. Rayleigh [31] suggested an approximate method for determining the natural frequencies of beams which

gives results sufficiently accurate for most purposes. This method equates the maximum potential energy to the maximum kinetic energy as determined from an assumed deflection curve. If the correct deflection curve is assumed, the exact answer for the fundamental frequency is obtained. The accuracy of the method, therefore, depends on how closely the deflection curve can be predicted.

When a beam is deflected potential energy is stored as a result of the bending. This energy may be expressed as the product of the average moment and the angle through which the beam is deflected, or

$$PE = \int \frac{M}{2}\, d\theta \qquad [7\cdot2]$$

where M = bending moment in inch-pounds
θ = angle of bending in radians

The equation for the angle of bending is

$$d\theta = \frac{M}{EI}\, dx$$

so that the potential energy may be written as

$$PE = \int_0^L \frac{M^2}{2EI}\, dx \qquad [7\cdot3]$$

The value of the bending moment is given by the equation

$$M = EI \frac{d^2y}{dx^2} \qquad [7\cdot4]$$

so that

$$PE = \frac{EI}{2}\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx \qquad [7\cdot5]$$

The kinetic energy is given by

$$KE = \tfrac{1}{2}mv^2 \qquad [7\cdot6]$$

The mass for a unit length beam is w/g; the velocity at any point with a deflection of y is

$$v = y\omega$$

The kinetic energy for the whole beam is found by integrating over the entire length of beam;

$$KE = \int_0^L \frac{1}{2}\frac{w}{g}\, dx(y\omega)^2 = \frac{w\omega^2}{2g}\int_0^L y^2\, dx \qquad [7\cdot7]$$

Since it is assumed that no energy is lost, the maximum kinetic energy as given by equation 7·7 is equal to the maximum potential energy as given by equation 7·5. If these two are equated and solved for ω^2, *the result for the lowest natural frequency is*

$$\omega^2 = \frac{EIg}{w} \frac{\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^L y^2\, dx} \qquad [7\cdot8]$$

where ω is in radians per second. The natural frequency is fortunately not particularly sensitive to the exact form of the deflection curve, and so, if an approximate value which fits the extreme conditions is assumed, a reasonably good approximation will result. If several deflection curves are tried the one giving the lowest frequency is the most accurate value.

Illustrative Problem. Find the natural frequency of a beam fixed at the ends as shown in Fig. 7·1.

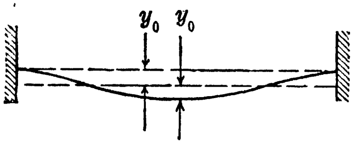


FIG. 7·1.

Solution. The beam deflects along the line indicated in Fig. 7·1. This line has approximately the form of a cosine wave with the axis shifted a distance equal to the amplitude of the cosine wave. Therefore, it would be reasonable to assume that the deflection equation is

$$y = y_0 - y_0 \cos\frac{x}{L}(2\pi)$$

Upon evaluating the integrals we have

$$\int_0^L \left(\frac{d^2y}{dx^2}\right)^2 dx = \int_0^L y_0^2 \left(\frac{2\pi}{L}\right)^4 \cos^2\frac{2\pi}{L}x\, dx = \frac{8\pi^4}{L^3} y_0^2$$

$$\int_0^L y_0^2 \left(1 - \cos\frac{x}{L} 2\pi\right)^2 dx = \frac{3Ly_0^2}{2}$$

When these values are substituted in equation 7·8 the frequency becomes

$$f^2 = \frac{\left(\frac{1}{2\pi}\right)^2 \frac{EIg}{w} \frac{y_0^2 8\pi^4}{L^3}}{\frac{3Ly_0^2}{2}} = \frac{4\pi^2}{3}\frac{EIg}{wL^4}$$

or

$$f = 3.63\sqrt{\frac{EIg}{wL^4}}$$

This result is very near the exact answer given in Table 7·2, where the constant is 3.57. The equation also shows that the assumed curve is not the exact one.

7·4. Energy Method. The solution of several types of problems is facilitated by utilizing the results of the energy method, particularly in dealing with distributed masses and multi-mass systems. The energy method also depends upon equating the potential and kinetic energies.

When a beam is deflected to its maximum position, all the energy stored in the system is potential energy and equal to

$$PE = \text{Average force (distance)}$$

If the beam is divided into many small parts, the weights of which are W, the total potential energy is equal to the sum of the potential energy of all the individual parts. The total deflection under a weight will be equal to the sum of the deflections at that point due to all the other individual weights. According to the theory of superposition, the reaction at a point due to several actions is equal to the sum of the reactions resulting from the application of each action taken individually. If the deflection from the equilibrium point is considered the change in potential energy for one weight W is

$$PE = \tfrac{1}{2}ky^2 \qquad [7\cdot 9]$$

where y is the displacement from the equilibrium position.

The maximum kinetic energy is given by the expression

$$KE = \frac{1}{2}\frac{W}{g}v^2 = \frac{1}{2}\frac{W}{g}y^2\omega^2 \qquad [7\cdot 10]$$

Since the kinetic energy is zero when the potential energy is a maximum, and vice versa, and the energy remains constant, the maximum total potential energy will equal the maximum total kinetic energy. Thus when the energies for all the individual masses are summed up we have

$$\tfrac{1}{2}\omega^2 \sum \frac{W}{g} y^2 = \tfrac{1}{2}\Sigma k y^2$$

or

$$\omega^2 = g\frac{\Sigma k y^2}{\Sigma W y^2} \qquad [7\cdot 11]$$

where ω = natural frequency in radians per second
g = 386 in./sec^2
k = spring constant at each weight in pounds per inch
W = increments of weight in pounds
y = deflection in inches
Σ = summation sign

If the deflection curve under dynamic action is the same as under the

static load, the dynamic deflections are proportional to the static deflections. Then

$$y = cy_s$$

and so equation 7·11 becomes

$$\omega^2 = g\frac{\Sigma ky^2}{\Sigma Wy^2} = \frac{g\Sigma ky_s^2}{\Sigma Wy_s^2}$$

As the value of k is W/y_s, the above expression may be reduced to

$$f = \frac{1}{2\pi}\sqrt{\frac{g\Sigma Wy_s}{\Sigma Wy_s^2}} \qquad [7\cdot 12]$$

This form reduces the work involved in numerical solutions since only the static deflection at each weight is needed.

Illustrative Problem. Find the natural frequency of a beam with a uniform section as shown in Fig. 7·2.

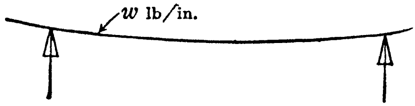


FIG. 7·2.

Solution. The deflection equation as given by textbooks on the strength of materials is

$$y = \frac{w}{24EI}(2Lx^3 - x^4 - L^3x)$$

where w = weight of unit length beam in pounds per inch. To find the values of ΣWy_s and ΣWy_s^2 for a uniform beam, it is necessary to integrate. These expressions are

$$\Sigma Wy_s = 2\int_0^{L/2}\frac{w}{24EI}(2Lx^3 - x^4 - L^3x)w\,dx = \frac{w^2L^5}{120EI}$$

$$\Sigma Wy_s^2 = 2\int_0^{L/2}\left(\frac{w}{24EI}\right)^2(2Lx^3 - x^4 - L^3x)^2w\,dx = \frac{31}{362{,}880}\frac{w^3L^9}{E^2I^2}$$

$w\,dx$ = weight for a length of shaft dx inches long

The natural frequency will be

$$f = \frac{1}{2\pi}\sqrt{\frac{gw^2L^5}{120EI}\frac{362{,}880E^2I^2}{31w^3L^9}}$$

$$= 1.57\sqrt{\frac{gEI}{wL^4}}\text{ radians/sec}$$

This result is the exact answer since the actual deflection curve is used.

7·5. Solution for Beam with Uniform Section. When the beam section is uniform the deflections may be obtained analytically by using the principles of superposition. This principle states that the total deflection caused by several weights is equal to the deflection caused by any one weight alone plus the deflections at that same point caused by each of the other weights acting individually. This holds within practical limits if the deflections do not become excessive. The method and all its details may best be explained by considering the following problem.

Illustrative Problem. A uniform shaft 48 in. long between bearings carries two pulleys, one weighing 50 lb and the other 35 lb. The large pulley is 12 in. from the left bearing, and the smaller pulley is 14 in. from the right bearing, as

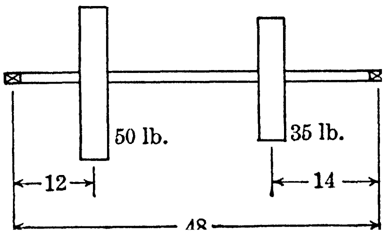


FIG. 7·3.

shown in Fig. 7·3. The shaft has a moment of inertia $I = 0.25$ in.^{4} and an $E = 30{,}000{,}000$ lb/in.^{2} The bearings are self-aligning so that the ends can be considered point supported. What is the lowest natural frequency for this system?

Solution. The deflections at the two points must be obtained first. They are obtained from the equation for the deflection due to a concentrated load, which is

$$y = \frac{Pbx}{6LEI}(L^2 - b^2 - x^2) \text{ for } x \leqq a$$

From this the deflection under the 50-lb load due to the 50-lb load is

$$y_1 = \frac{Pba}{6LEI}(L^2 - b^2 - a^2) = \frac{Pa^2b^2}{3EIL}$$

$$= \frac{50(12^2)36^2}{3(30)10^6(0.25)48} = 0.00830 \text{ in.}$$

At the 35-lb load due to the 35-lb load the deflection is

$$y_2 = \frac{35(14^2)34^2}{3(30)10^6(0.25)48} = 0.00733 \text{ in.}$$

The deflection at the 50-lb load due to the 35-lb load is

$$y_3 = \frac{35(14)12(48^2 - 14^2 - 12^2)}{6(30)10^6(0.25)48} = 0.00533 \text{ in.}$$

The deflection at the 35-lb load due to the 50-lb load is

$$y_4 = \frac{50(12)14(48^2 - 12^2 - 14^2)}{6(30)10^6(0.25)48} = 0.00762 \text{ in.}$$

The total deflections at the 50-lb load and 35-lb load are

$$y_{50} = y_1 + y_3 = 0.00830 + 0.00533 = 0.01363 \text{ in.}$$

$$y_{35} = y_2 + y_4 = 0.00733 + 0.00762 = 0.01495 \text{ in.}$$

The natural frequency then

$$f = \frac{1}{2\pi}\sqrt{\frac{g(W_{50}y_{50} + W_{35}y_{35})}{W_{50}y_{50}^2 + W_{35}y_{35}^2}} \qquad [7\cdot 13]$$

$$= \frac{1}{2\pi}\sqrt{\frac{g[50(0.01363) + 35(0.01495)]}{50(0.01363)^2 + 35(0.01495)^2}}$$

$$= 26.3 \text{ cycles/sec or } 1575 \text{ cycles/min}$$

Among the several approximations that have been considered here are that the hubs of the pulleys have no influence on the shaft stiffness and that any belts present have no influence on the system. It can be proved that the natural frequencies for lateral vibration are the same as the critical speeds for a rotor. This would indicate that a speed of 1575 rpm would not be advisable. Other critical speeds above this fundamental frequency must be determined by other means.

7·6. Solution for Beam with Non-uniform Section. When the shaft is not of uniform section, the determination of a formula for the deflection becomes impractical because the value of the sectional moment of inertia as well as of the bending moment varies with the distance along the shaft. The basic equation involving deflection is

$$\frac{d^2y}{dx^2} = \frac{-M}{EI} \qquad [7\cdot 14]$$

If the quantity M/EI is plotted and graphically integrated twice, the deflection curve is obtained. Several methods are available for performing this integration. The method used here is the complete graphical method. After the deflections are known, equation 7·12 may be used to determine the natural frequency, that is,

$$f = \frac{1}{2\pi}\sqrt{\frac{g\Sigma W y_s}{\Sigma W y_s^2}}$$

A specific problem illustrates the procedure.

Illustrative Problem. A steel shaft has two forces applied as shown in Fig. 7·4. The dimensions are as shown. What is the fundamental natural frequency?

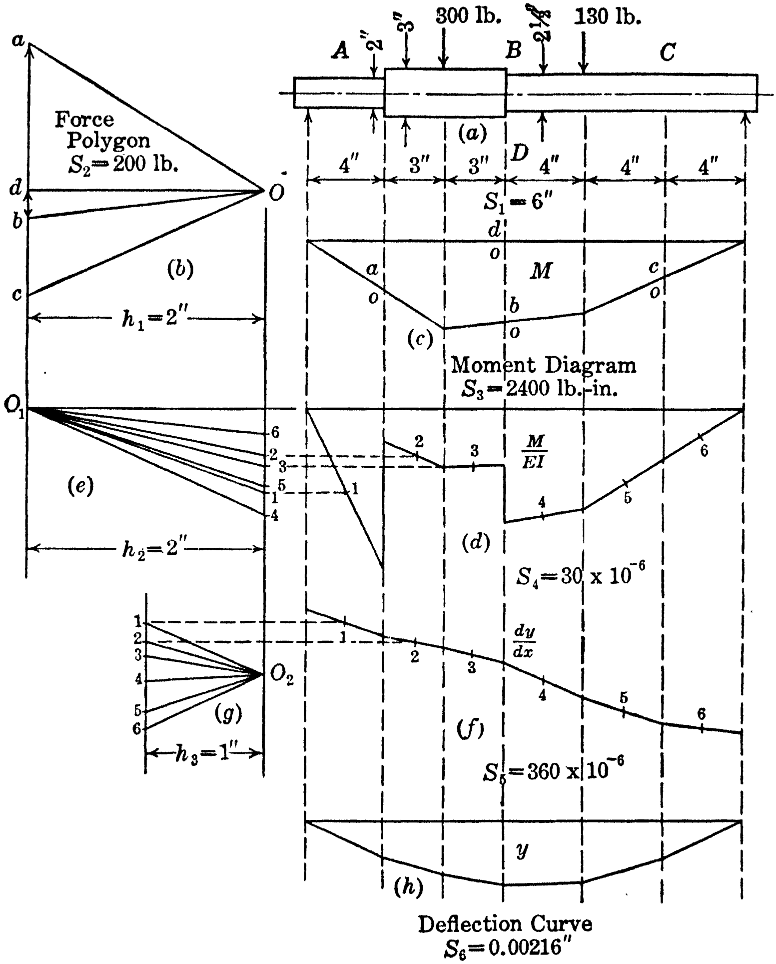


FIG. 7·4.

Solution. After the shaft has been laid out to some scale, Fig. 7·4*a*, and the loads put in position, a force polygon, Fig. 7·4*b*, should be constructed using Bow's notation for funicular force polygons. (The force between space *A* and space *B* is force *ab*, etc.) After some arbitrary point *O* is chosen at a convenient distance h_1 from the forces, the force rays *aO*, *bO*, etc., are drawn in. At the same time, lines parallel to *aO*, *bO*, etc., are drawn between the lines of action of the original space drawing beginning with *aO* across space *A*, that is, from

the line of action of DA to the line of action of AB. After cO is drawn, one final line is drawn in across space D in such a way as to intersect the original starting point of aO. This line closes the funicular polygon. Now by drawing dO in Fig. 7·4b parallel to dO of Fig. 7·4c, the point d of the force polygon is determined, and thereby the two reactions of the bearings on the shaft can be scaled off. At the same time the vertical distances between lines of the polygon in Fig. 7·4c represent the bending moments at the corresponding positions on the space drawing. The scale is equal to $S_3 = S_1S_2h_1$, which in this example is $S_3 = 6(200)2 = 2400$ lb-in. All scales S are the values for one inch. Under the 300-lb load the moment is approximately $2400(0.73) = 1750$ lb-in.

The remaining moments may be determined in like manner. Now the values of M/EI must be determined and plotted as shown in Fig. 7·4d. The length of the shaft has been broken up into several parts. It is necessary to evaluate M/EI for every point where the section changes or a force is applied. For succeeding work the accuracy depends upon the number of increments into which the beam is divided. In this case it seems advisable to divide up the space between the 130-lb load and the right reaction. The plot of M/EI is shown in Fig. 7·4d to a scale of $S_4 = 1$ in. $= 30(10^{-6})$. After the midpoints of the chosen sections of the shaft are selected and then projected across to a vertical line, rays are again drawn from these latter points to a pole point O_1 located on the same line as the base line for the M/EI diagram, as shown in Fig. 7·4e. At the same time lines are drawn successively across the chosen sections parallel to the ray corresponding to the midpoint of that section. Figure 7·4f is the resulting diagram that represents the relative slope. The scale is $S_5 = S_4S_1h_2 = 30(10^{-6})6(2) = 360(10^{-6})$. By taking the midpoints again and drawing the rays, the deflection curve shown in Fig. 7·4h is obtained. Any convenient pole point may be used in Fig. 7·4g in order to get a well-balanced deflection curve. At the same time the pole has been taken to the right of the vertical line to obtain the deflection downward. The deflection scale will be $S_6 = S_5S_1h_3 = 360(10^{-6})6(1) = 2160(10^{-6}) = 0.00216$ in./in. We can see that if the pole distance had been taken as 0.926 in. instead of 1 in. the deflection scale would have been 0.002 in./in. From the end conditions we know that the zero deflection line will extend from the points where the first and last rays intersect the lines of action of the reactions. After drawing this zero deflection line, any deflection along the beam may be determined by measuring the vertical distance between the zero line and the deflection curve. The maximum deflection is $y_{\max} = 0.00216(0.53) = 0.00114$ in. The deflection under the 130-lb load is 0.00108 in. and the deflection under the 300-lb load is 0.00095 in. From this the natural frequency may be determined as being

$$f = \frac{1}{2\pi}\sqrt{\frac{g(W_1y_1 + W_2y_2)}{W_1{y_1}^2 + W_2{y_2}^2}} \qquad [7\cdot15]$$

$$f = \frac{1}{2\pi}\sqrt{\frac{386[300(0.000950) + 130(0.00108)]}{300(0.000950)^2 + 130(0.00108)^2}}$$

$$= 99.2 \text{ cycles/sec or } 5950 \text{ cycles/min}$$

This indicates the lowest critical speed would be around 5950 rpm and should, therefore, be avoided. The influence of hubs and such may be considered by using the hub diameter as part of the shaft. The flexibility of the bearings and supports has been neglected in all cases here. Where flexibility is present in the supports the frequencies are lowered considerably, many times to less than three-fourths of the calculated value.

7·7. General Solution for Beam Vibration. The general analysis of beam vibration requires an equation for the beam bending moment as expressed in terms of the physical properties. This relation as given in any textbook on strength of materials can be written

$$M = EI \frac{d^2y}{dx^2} \qquad [7\cdot16]$$

where M = bending moment in inch-pounds
E = modulus of elasticity in pounds per square inch
I = section moment of inertia in inches^4
y = beam deflection in inches
x = distance along the beam in inches

Another equation giving the relation between shear force, V, and the change of the bending moment along the beam is

$$V = \frac{dM}{dx} \qquad [7\cdot17]$$

where V = shear force in pounds

A vibrating beam may be assumed to have simple harmonic motion at any one point. The acceleration of any point may then be written as

$$A = y\omega^2$$

where A = acceleration in inches per second per second
y = deflection in inches
ω = angular frequency of vibration in radians per second

If the weight per inch of the beam is w, the force can be written as

$$dF = (dm)A = \left(\frac{w}{g}\right)(y\omega^2)\,dx$$

or

$$\frac{dF}{dx} = \frac{w}{g}\,y\omega^2 \qquad [7\cdot18]$$

That is, the change in vertical dynamic load over a length dx of beam is

given as $(w/g)y\omega^2$. From equations 7·16 and 7·17 it is found that the change in vertical shear force is

$$\frac{dV}{dx} = \frac{d^2M}{dx^2} = \frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right)$$

The dynamic load dF/dx on the vibrating beam acts as a beam load dV/dx, and so the equation of motion for a uniform beam is

$$EI\frac{d^4y}{dx^4} = \frac{w}{g}\omega^2 y$$

or

$$\frac{d^4y}{dx^4} - \frac{w}{g}\frac{\omega^2}{EI}y = 0 \qquad [7\cdot 19]$$

Equation 7·19, though similar in form to those discussed in Chapter II, involves the fourth derivative of y. As in the equations discussed in Chapter II, the solution of y in terms of x must be such that when differentiated it will not change its general form. As in those examples the solution must have the form

$$y = C\sin mx, \quad C\cos mx, \quad \text{or} \quad Ce^{mx} \qquad [7\cdot 20]$$

Since Ce^{mx} is the most convenient form it will be used here. The solution of equation 7·19 can be obtained by the method outlined in section 2·3. The auxiliary equation is

$$m^4 - \frac{w}{g}\frac{\omega^2}{EI} = 0 \qquad [7\cdot 21]$$

The roots of this equation are

$$m_1 = p \qquad m_2 = -p$$
$$m_3 = ip \qquad m_4 = -ip \qquad [7\cdot 22a]$$

where

$$p = \sqrt[4]{\frac{w}{g}\frac{\omega^2}{EI}} \qquad [7\cdot 22b]$$

The general solution is, therefore,

$$y = C_1e^{px} + C_2e^{-px} + C_3e^{ipx} + C_4e^{-ipx} \qquad [7\cdot 23]$$

In section 2·3 it was shown that by making the substitution

$$e^{ipx} = \cos px + i\sin px$$

and

$$e^{-ipx} = \cos px - i\sin px$$

the last two terms of equation 7·23 could be written in terms of $\cos px$ and $\sin px$, giving

$$y = C_1 e^{px} + C_2 e^{-px} + A_3 \sin px + A_4 \cos px \qquad [7\cdot 24a]$$

Similarly, it can be shown that

$$e^{px} = \cosh px + i \sinh px$$

$$e^{-px} = \cosh px - i \sinh px$$

so we can write

$$C_1 e^{px} + C_2 e^{-px} = A_1 \sinh px + A_2 \cosh px$$

The resulting form of equation 7·23 would be

$$y = A_1 \sinh px + A_2 \cosh px + A_3 \sin px + A_4 \cos px \qquad [7\cdot 24b]$$

Equation 7·24*b* will represent the deflection curve of any beam having a uniform section. The values of the constants C and A will depend upon the boundary conditions as determined by the type of beam and the manner in which it is supported. These boundary conditions are expressed as deflections, slopes, bending moments, or shear forces. At each end of the beam two of these quantities are zero. For example, $y = 0$ expresses the condition that the deflection is zero, $dy/dx = 0$ means that the slope of the beam is zero, $d^2y/dx^2 = 0$ indicates that the bending moment is zero, and $d^3y/dx^3 = 0$ shows that the shear force is zero. It is also necessary to specify the points in the beam where values are zero.

Illustrative Problem. Find the natural frequency of a uniform beam simply supported at the ends.

Solution. The end conditions which apply are $y = 0$ and $d^2y/dx^2 = 0$ at both $x = 0$ and $x = L$. If the conditions $y = 0$ at $x = 0$ are substituted in equation 7·24*b*, we find

$$0 = A_1(0) + A_2(1) + A_3(0) + A_4(1)$$

or

$$A_2 + A_4 = 0 \qquad [7\cdot 25a]$$

The condition $d^2y/dx^2 = 0$ at $x = 0$ will give

$$\frac{dy}{dx} = pA_1 \cosh px + pA_2 \sinh px + pA_3 \cos px - pA_4 \sin px$$

$$\frac{d^2y}{dx^2} = p^2A_1 \sinh px + p^2A_2 \cosh px - p^2A_3 \sin px - p^2A_4 \cos px$$

$$0 = p^2A_1(0) + p^2A_2(1) - p^2A_3(0) - p^2A_4(1) \qquad [7\cdot 25b]$$

or

$$A_2 - A_4 = 0$$

Adding, and subtracting this value from $7 \cdot 25a$ gives

$$A_2 = A_4 = 0$$

The solution then reduces to

$$y = A_1 \sinh px + A_3 \sin px \qquad [7\cdot 25c]$$

Using the condition $y = 0$ at $x = L$ we have

$$0 = A_1 \sinh pL + A_3 \sin pL \qquad [7\cdot 25d]$$

The condition $d^2y/dx^2 = 0$ at $x = L$ gives

$$0 = p^2 A_1 \sinh pL - p^2 A_3 \sin pL \qquad [7\cdot 25e]$$

Adding, we have

$$A_1 \sinh pL = 0 \qquad [7\cdot 25f]$$

Subtraction gives

$$A_3 \sin pL = 0 \qquad [7\cdot 25g]$$

Since $\sinh pL$ is zero only for $L = 0$, A_1 must be zero to make the term $A_1 \sinh pL$ zero for any other values of L. However, the value of $\sin pL$ can equal zero for many values of pL. Then, if y is not zero at all places, A_3 cannot be zero. The solution will be

$$y = A_3 \sin px \qquad [7\cdot 26]$$

The deflection is always zero at $x = L$ regardless of the mode of vibration; therefore we can say that

$$pL = \pi, \qquad 2\pi, \qquad 3\pi \cdots n\pi$$

or that

$$p = \frac{n\pi}{L}$$

Upon equating this value of p with that in equation $7\cdot 22b$ we have

$$\frac{n\pi}{L} = \sqrt[4]{\frac{w}{g}\frac{\omega^2}{EI}}$$

or

$$\omega = \frac{n^2\pi^2}{L^2}\sqrt{\frac{gEI}{w}}$$

This gives

$$f = \frac{n^2\pi^2}{2\pi}\sqrt{\frac{gEI}{wL^4}} = C\sqrt{\frac{gEI}{wL^4}} \text{ cycles/sec} \qquad [7\cdot 27]$$

where $C = \frac{n^2\pi}{2} = \frac{\pi}{2}, \frac{4\pi}{2}, \frac{9\pi}{2}$, etc.

g = 386 in./sec^2
E = modulus of elasticity in pounds per square inch
I = moment of inertia of beam section in inches^4
w = beam weight in pounds per inch
L = beam length in inches

This equation shows that the natural frequencies of a uniform beam simply sup-

ported at the ends progress as 1, 4, 9, 16, etc. The modes of vibration for these frequencies are indicated in Table 7·2. Modes are given at certain fractions of the beam length measured from the right end. Equation 7·27 although derived for a particular problem is actually a general form for any uniform beam. The values of C for any other beam are given in Table 7·2.

7·8. Effect of Distributed Weight. When beams carry a concentrated weight the energy method can be used for determining the effect of the weight of the beam. It is convenient to replace the distributed beam weight by an equivalent weight placed at the concentrated weight. Then the problem is solved by the method explained in section 7·2 for a concentrated weight on a beam. Dent [32] has worked out a number of these beam problems. The method may best be described by following through an illustrative problem.

Illustrative Problem. Find the proportion of the beam weight that must be added to the concentrated weight of Fig. 7·5 to allow for its distributed weight.

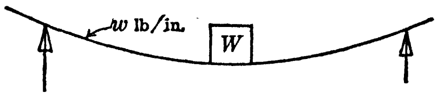


FIG. 7·5.

Solution. By equation 7·12 the frequency is

$$f = \frac{1}{2\pi}\sqrt{\frac{g\Sigma Wy}{\Sigma Wy^2}} \text{ cycles/sec}$$

If it assumed that part of the distributed weight is combined with the concentrated weight so that the composite weight at the end is W', the equation is

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{y_s}} = \frac{1}{2\pi}\sqrt{\frac{g48EI}{W'L^3}}$$

The conditions expressed by these equations are

$$\frac{1}{y_s} = \frac{\Sigma Wy}{\Sigma Wy^2} = \frac{48EI}{W'L^3}$$

or

$$W' = \frac{48EI}{L^3}\frac{\Sigma Wy^2}{\Sigma Wy}$$

If it is assumed that the total deflection is given by the deflection due to weight W plus the deflection by the distributed w we have

$$y = y_1 + y_2 = \frac{W}{48EI}(4x^3 - 3L^2x) + \frac{w}{24EI}(2Lx^3 - x^4 - L^3x)$$

At midspan, where the concentrated weight is, the deflection is

$$y_0 = \frac{WL^3}{384EI}(5wL + 8W)$$

TABLE 7·2.

Cantilever	$C=.560$	.226 $C=3.51$	.4999 .132 $C=9.82$	.644 .356 .094 $C=19.2$	.721 .500 .277 .0735 $C=31.8$
Simply Supported Ends	$C=1.57$	.500 $C=6.28$	.333 .333 $C=14.1$	.75 .50 .25 $C=25.2$	.80 .60 .40 .20 $C=39.4$
Fixed Ends	$C=3.56$	.500 $C=9.82$	.641 .359 $C=19.2$	.722 .5 .278 $C=31.8$	.773 .591 .409 .227 $C=47.5$
Free Ends	.776 .224 $C=3.56$	.868 .500 .132 $C=9.82$	.906 .644 .356 .094 $C=19.2$	.927 .723 .500 .277 .0734 $C=31.8$	.940 .774 .591 .409 .227 .060 $C=47.5$
Fixed-Hinged	$C=2.45$	.440 $C=7.95$	.616 .308 $C=16.6$	.706 .471 .235 $C=28.4$	.762 .571 .381 .190 $C=43.3$
Hinged-Free	.264 $C=2.45$	.554 .137 $C=7.95$	.692 .384 .102 $C=16.6$	.765 .529 .293 .0777 $C=28.4$	.810 .619 .429 .237 .0630 $C=43.3$

$$f_n = C\sqrt{\frac{gEI}{wL^4}}$$

f_n = natural frequency in cycles/sec.
C = constant from above table
g = 386 in. sec.^{2}
E = Modulus of Elasticity in lb./in.^{2}

I = Sectional moment of inertia in in.^{4}
w = Weight of unit length beam in lb./in.
L = Beam Length in inches

For concentrated loads

$$Wy_0 = \frac{W^2L^3}{384EI}(5wL + 8W)$$

$$Wy_0^2 = W\left(\frac{WL^3}{384EI}\right)^2(5wL + 8W)^2$$

For the distributed load

$$\Sigma Wy_d = 2\int_0^{L/2} w(y_1 + y_2)\,dx = \frac{w}{1920EI}(16wL^5 + 25WL^4)$$

$$\Sigma Wy_d^2 = 2\int_0^{L/2} w(y_1 + y_2)^2\,dx = \frac{2w}{(48EI)^2}\left[\frac{62w^2L^9}{630} + \frac{277WwL^8}{896} + \frac{17W^2L^7}{70}\right]$$

The total values of the summations are

$$Wy_0 + \Sigma Wy_d = \frac{L^3}{1920EI}(40W^2 + 50WwL + 16w^2L^2)$$

$$Wy_0^2 + \Sigma Wy_d^2 =$$

$$\frac{L^3}{315(384EI)^2}(20{,}160W^3 + 33{,}912W^2wL + 20{,}340Ww^2L^2 + 3968w^3L^3)$$

If $K = wL/W$, and the last two equations are used to solve for W' as previously indicated, the result is

$$W' = W + W_b\left(\frac{2178 + 3069K + 992K^2}{5040 + 6300K + 2016K^2}\right) = W + UW_b \qquad [7\cdot28]$$

Where $W_b = wL$ the value of the term in parentheses or U is the fraction of the beam weight desired. K is the ratio of the beam weight to the concentrated weight. The value of U for any value of K is indicated in Fig. 7·6. This figure

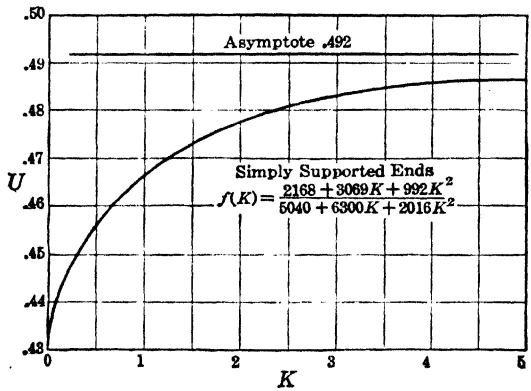


Fig. 7·6.

shows that the variation ranges from 0.432 at $K = 0$ to 0.492 at $K = \infty$. An average value of about 0.46 would seem reasonable for most beam problems.

A similar analysis shows that for a cantilever with a weight at the end

$$U = \frac{5940 + 4410K + 910K^2}{25{,}200 + 18{,}900K + 3780K^2} \qquad [7\cdot29]$$

The values for this are shown in Fig. 7·7. A value of 0.24 will be close enough for all practical problems.

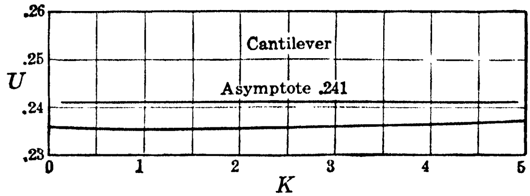


FIG. 7·7.

For a beam with ends fixed and a weight in the middle the constant becomes

$$U = \frac{233 + 219K + 64K^2}{630 + 630K + 168K^2} \qquad [7\cdot30]$$

The values of U are indicated in Fig. 7·8. Oddly it reaches a minimum of about 0.361 at $K = 1.4$. A reasonable average would seem to be 0.37 or $\frac{3}{8}$.

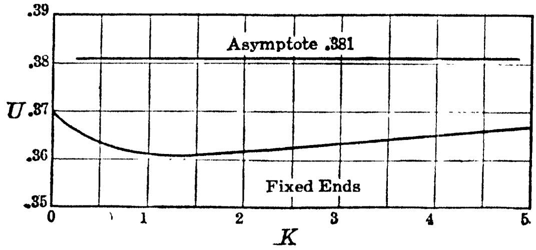


FIG. 7·8.

Professor Dent has also prepared tables for:

A cantilever beam with load not at end of beam (Table 7·3).
A simple beam carrying a single load (Table 7·4).
A fixed beam with a simple load (Table 7·5).
A simple beam with two loads (Table 7·6).
A fixed beam with two loads (Table 7·7).

K and N as used in the tables are defined as

$$K = \frac{\text{Total distributed load}}{\text{Concentrated load}}$$

$$n = \frac{\text{Distance of concentrated load from end}}{\text{Length of beam}}$$

The values given in Tables 7·3 to 7·7 are the ratio of equivalent single concentrated load to total load, P. The equivalent load for a cantilever beam is considered concentrated at the end and for the simple and fixed beams at the center.

TABLE 7·3

K	$n = 0.2$	$n = 0.4$	$n = 0.6$	$n = 0.8$	$n = 1.0$
0.0	0.008	0.064	0.216	0.512	1.000
0.1	0.022	0.077	0.218	0.486	0.925
0.2	0.038	0.088	0.219	0.466	0.874
0.5	0.078	0.115	0.222	0.421	0.746
1.0	0.120	0.145	0.226	0.376	0.618
2.0	0.160	0.176	0.230	0.331	0.489
5.0	0.201	0.209	0.235	0.285	0.362
10.0	0.219	0.223	0.238	0.264	0.307
100.0	0.238	0.238	0.241	0.243	0.248

TABLE 7·4

K	$n = 0.1$ or 0.9	$n = 0.2$ or 0.8	$n = 0.3$ or 0.7	$n = 0.4$ or 0.6	$n = 0.5$
0.0	0.130	0.410	0.706	0.923	1.000
0.1	0.153	0.411	0.686	0.907	0.950
0.2	0.176	0.412	0.669	0.887	0.914
0.5	0.232	0.419	0.628	0.825	0.830
1.0	0.293	0.432	0.587	0.742	0.741
2.0	0.361	0.448	0.552	0.648	0.658
5.0	0.426	0.468	0.519	0.556	0.576
10.0	0.457	0.478	0.506	0.524	0.537
100.0	0.491	0.490	0.494	0.498	0.499

TABLE 7·5

K	$n = 0.1$ or 0.9	$n = 0.2$ or 0.8	$n = 0.25$ or 0.75	$n = 0.3$ or 0.7	$n = 0.4$ or 0.6	$n = 0.5$
0.0	0.047	0.263	0.420	0.592	0.895	1.000
0.1	0.058	0.261	0.406	0.567	0.836	0.942
0.2	0.074	0.261	0.396	0.546	0.795	0.896
0.5	0.125	0.269	0.377	0.501	0.710	0.791
1.0	0.188	0.287	0.363	0.463	0.621	0.689
2.0	0.253	0.314	0.362	0.428	0.538	0.584
5.0	0.319	0.347	0.367	0.400	0.456	0.480
10.0	0.348	0.362	0.373	0.391	0.420	0.433
100.0	0.378	0.380	0.380	0.383	0.387	0.388

TABLE 7·6

K	$n = 0.1$ or 0.9	$n = 0.2$ or 0.8	$n = 0.25$ or 0.75	$n = 0.3$ or 0.7	$n = 0.4$ or 0.6	$n = 0.5$
0.0	0.104	0.352	0.500	0.648	0.897	1.000
0.1	0.138	0.363	0.496	0.637	0.861	0.950
0.2	0.170	0.374	0.492	0.626	0.827	0.914
0.5	0.234	0.398		0.607	0.762	0.830
1.0	0.299	0.423	Con-	0.582	0.692	0.741
2.0	0.363	0.448	stant	0.556	0.626	0.658
5.0	0.426	0.472	at	0.526	0.559	0.576
10.0	0.456	0.481	491	0.510	0.529	0.537
100.0	0.488	0.491		0.494	0.497	0.499

TABLE 7·7

K	$n = 0.1$ or 0.9	$n = 0.2$ or 0.8	$n = 0.25$ or 0.75	$n = 0.3$ or 0.7	$n = 0.4$ or 0.6	$n = 0.5$
0.0	0.027	0.179	0.312	0.474	0.819	1.000
0.1	0.048	0.192	0.316	0.465	0.782	0.942
0.2	0.071	0.204	0.320	0.456	0.747	0.896
0.5	0.131	0.233	0.329	0.443	0.675	0.791
1.0	0.194	0.266	0.340	0.425	0.600	0.689
2.0	0.257	0.302	0.351	0.410	0.526	0.584
5.0	0.320	0.340	0.366	0.394	0.453	0.480
10.0	0.355	0.358	0.372	0.387	0.420	0.433
100.0	0.377	0.379	0.380	0.382	0.385	0.388

Illustrative Problem. A simply supported 10-in. I beam 10 ft long weighs 30 lb/ft, carries two concentrated loads of 1000 lb each 3 ft from the ends, and an additional distributed load of 70 lb/ft. Find its natural frequency. (I = 133.5)

Solution.

$$K = \frac{(30 + 70)(10)}{2000} = 0.5$$

$$N = \tfrac{3}{10} = 0.3$$

From Table 7·6 we find for $K = 0.5$ and $N = 0.3$ and 0.7 that $P = 0.607$. The equivalent concentrated load at center = (2000 + 1000)(0.607) = 1821. From equation 7·1 and Table 7·1

$$f = \frac{1}{2\pi}\sqrt{\frac{48(30)(10^6)(386)(133.5)}{(1821)(120^3)}} = 21.3 \text{ cycles/sec}$$

7·9. Effect of Elastic Support. All previous examples have assumed that the supporting structure is rigid or has no motion. This assumption is incorrect because the support always moves to some degree, and so the natural frequency is lower than the calculations would indicate. The error might be estimated by providing for an increased elasticity. The effect might be visualized better by referring to Fig. 7·8, where the shaft is extended to intersect the zero axis when the beam is in equilibrium. The effective length of the shaft is therefore longer than the actual length. The effective length would be equal to L_e in Fig. 7·9.

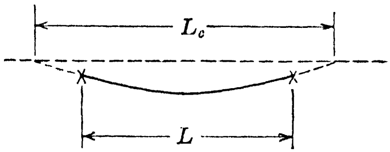


FIG. 7·9.

7·10. Vibration of Thin Flat Plates. The theory of vibration of plates is rather difficult, and the results are of limited value in solving problems. The usual equation for the frequency of a rectangular plate with simply supported edges is given as [33]

$$f_n = \frac{\pi}{2}\sqrt{\frac{gEt^2}{w12(1-\mu^2)}}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \text{ cycles/sec} \qquad [7\cdot31]$$

where a = width of plate in inches

b = length of plate in inches

E = modulus of elasticity in pounds per square inch

t = plate thickness in inches
w = weight density in pounds per cubic inch
μ = Poisson's ratio—usually 0.3
m and n = integers describing mode of vibration
$m = n = 1$ for lowest frequency

The next frequency will be found by taking m or n equal to 2 while the other is taken as 1. The general shapes for the lowest four frequencies are shown in Fig. 7·10. These views furnish a visual picture of how

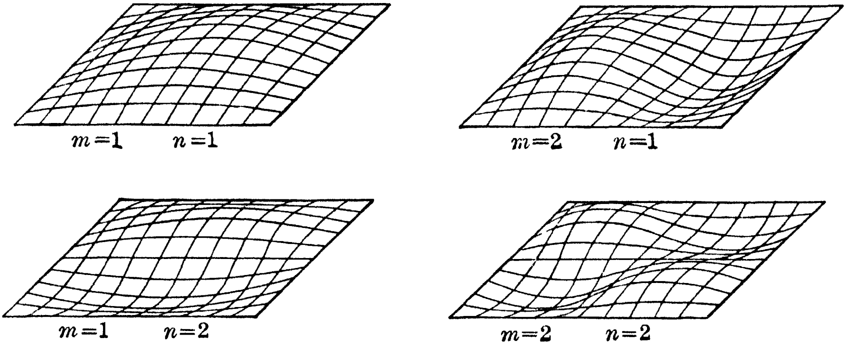


FIG. 7·10.

plates tend to vibrate. They show why stiffeners dividing a vibrating panel symmetrically may not change the natural frequency of the panels.

PROBLEMS

7·1. Find the fundamental frequency for the system in Fig. P7·1. The deflections of the weights are as listed.

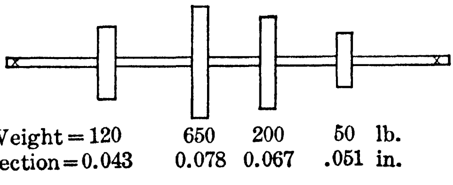


FIG. P7·1.

7·2. A steel beam 12 ft long has two weights mounted upon it. The beam is simply supported at the ends. One weight is 3 ft from one end and weighs 650 lb; the other weighing 1200 lb is 4 ft 6 in. from the other end. The beam has a uniform sectional moment of inertia $I = 64.5$ in.^{4} Since the structure is subject to considerable vibrations, determine the natural frequency. The weight of the beam may be neglected.

7·3. Find the fundamental frequency for the shaft loaded as indicated in Fig. P7·3.

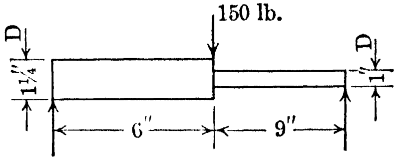


FIG. P7·3.

7·4. Find the lowest natural frequency for the shaft loaded as shown in Fig. P7·4.

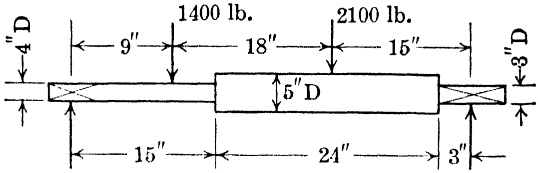


FIG. P7·4.

7·5. Determine the fundamental frequency for the stepped shaft shown in Fig. P7·5.

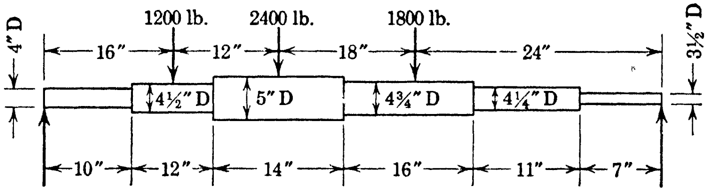


FIG. P7·5

7·6. A steel cantilever beam ($E = 30{,}000{,}000$) $\frac{1}{4}$ in. deep and 1 in. wide supports a load of 9.8 lb at its free end which is 18 in. from the fixed end. What is the natural frequency, if the weight of the beam is neglected? Steel weighs 0.283 lb/in.^{3}

7·7. If the cantilever weight is considered in problem 7·6, what is the natural frequency? What is the percentage error if the weight of the bar is neglected? Work using method of section 7·4.

7·8. Solve problem 7·7 using the method of section 7·8.

7·9. What fraction of the concentrated weight may a bar weigh and still have an error of only 10 per cent for a uniform beam simply supported?

7·10. Solve problem 7·3 for a cantilever.

7·11. Using the Rayleigh equation 7·8, derive an expression for the natural frequency of a uniform beam simply supported at the ends.

7·12. A beam with a sectional moment of inertia of 0.67 supports three weights, 3, 4.7, and 6 lb, whose static deflections are 0.21, 0.34, and 0.29 in., respectively. What is the lowest natural frequency of this system?

7·13. Using the general method outlined in section 7·7 derive the general expression for natural frequency for a cantilever beam of uniform section.

7·14. Verify the values of C as given in Table 7·2 for a beam with fixed ends.

7·15. An autogyro rotor blade is hinged at the inner end. The blade is 20 ft long and weighs 60 lb. If the mean moment of inertia of the blade section is 0.143, determine the natural frequency neglecting the centrifugal force.

7·16. A structural member shown in Fig. P7·16 has plates fastened to the upper

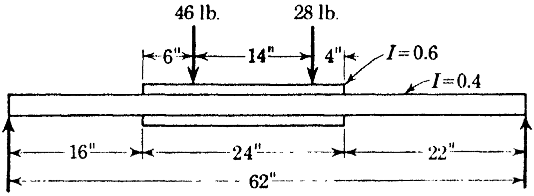


Fig. P7·16.

and lower sides near the mid-portion of its length. The moments of inertia are as given, and two loads are applied as shown. What is the fundamental frequency?

7·17. Find the lowest natural frequency for the steel shaft loaded as shown in Fig. P7·17.

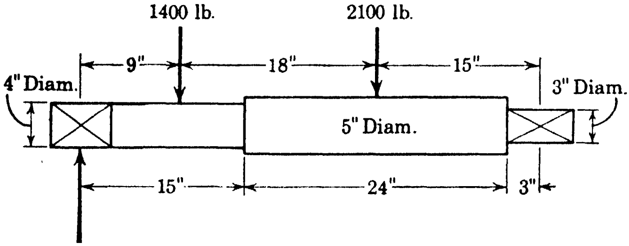


Fig. P7·17.

CHAPTER VIII

SOUND

8·1. Introduction. Sound is a form of vibration which is audible. Music is that portion of sound which is pleasing to the ear while noise is the unpleasant portion. Engineers are interested professionally mostly in the latter because of its influence on people as well as on a machine or structure. It is quite well established that noise affects the performance and comfort of people in the factory, office, and home. Excessive noise in a machine may be an indication of poor balance, excessive clearance, turbulent flow or some other factor affecting the life of the equipment. Manufacturers of most types of machines are now very noise-conscious because customers are demanding quieter performance. Equipment with a low noise level has a sales advantage. Therefore, it is necessary for engineers, designers, and salesmen to know more about noise and its reduction. This chapter is intended to give the more basic knowledge necessary to understand the problems encountered in noise levels and their reduction.[34,35,36]

8·2. Sound Terminology. It is necessary to know the definition of certain terms commonly used in sound technology. Most common is the decibel.[37,38]

The *decibel* is $\frac{1}{10}$ of a bel which is the fundamental division of a logarithmic scale which expresses the ratio of two amounts of power. It is given by the expression

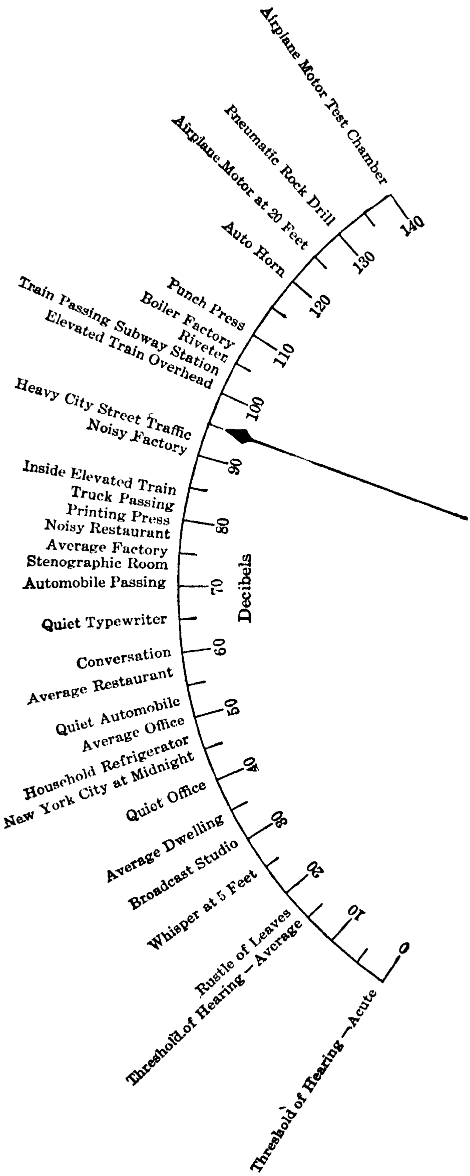
$$\text{db} = 10 \log_{10} \frac{P_1}{P_0} \qquad [8\cdot 1]$$

where db = decibels
P = power

If the power varies as the sound intensity, the expression is

$$\text{db} = 10 \log_{10} \frac{I_1}{I_0} \qquad [8\cdot 2]$$

Sound intensity is the sound energy transmitted per unit time in a specified direction through a unit area normal to this direction at the selected point. Units are ergs per second per square centimeter or watts per square centimeter.



Courtesy General Radio Co.

FIG. 8·1.

The *intensity level* in decibels if given by equation 8·2 when the base intensity $I_0 = 10^{-16}$ watt/cm^2. Figure 8·1 shows some of the more common average noise levels.

The *effective sound pressure* at a point is the root-mean-square value of the instantaneous sound pressure over a complete cycle. It is commonly called *sound pressure*.

The *pressure level* in decibels is given by the equation

$$\text{db} = 20 \log_{10} \frac{P_1}{P_0} \qquad [8\cdot3]$$

where $P_0 = 0.0002$ dynes/cm^2. This definition assumes that the power varies as the square of the sound pressure.

The *velocity level* in decibels is given by the expression

$$\text{db} = 20 \log_{10} \frac{V_1}{V_0} \qquad [8\cdot4]$$

where $V_0 = 5 \times 10^{-16}$ cm/sec. This also assumes that the power varies as the square of the particle velocity.

Acoustic absorption is the process by which a portion of the sound which strikes a surface is absorbed and does not reflect back into space. A *sabin* is a unit of equivalent absorption and is equal to a square foot of total absorption. This means that 2 sq ft of surface which absorb half of the incident sound energy have one sabin absorption.

Reverberation is the persistence of sound due to repeated reflections. The *reverberation time* for a given frequency is the time in seconds required for the average sound energy density to decrease (after the source is stopped) to one millionth of its initial steady state value. This is equivalent to a drop of 60 db.

Loudness is that subjective quality of sound which determines the magnitude of the human auditory sensation produced by that sound. The *loudness level* of any sound in phons is judged by listeners to be equivalent in loudness to a pure tone of 1000 cycles/sec and is numerically equal to the decibel reading of this tone. The weighting scales on the sound level meters are designed to give an approximate phon reading and as such do not follow any equation for decibels. Thus, in Fig. 8·2 the normal relation between decibels and phons shows that frequency affects the loudness level. The contour lines indicate what the ear hears as equal loudness. Figure 8·3 shows the relation between the loudness level and the loudness. From this it can be seen that a reduction of level from 72 phons to 40 phons gives a noise one-tenth as loud.

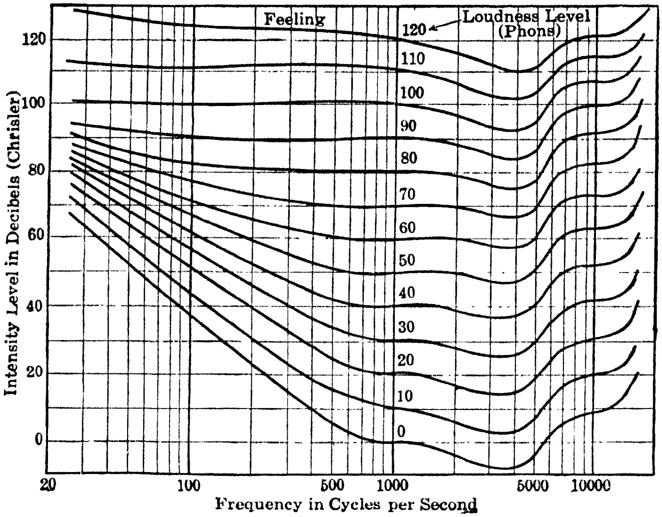

Feeling
Loudness Level (Phons)
120
110
100
90
80
70
60
50
40
30
20
10
0
Intensity Level in Decibels (Chrisler)
Frequency in Cycles per Second
20
100
500
1000
5000
10000

FIG. 8·2.

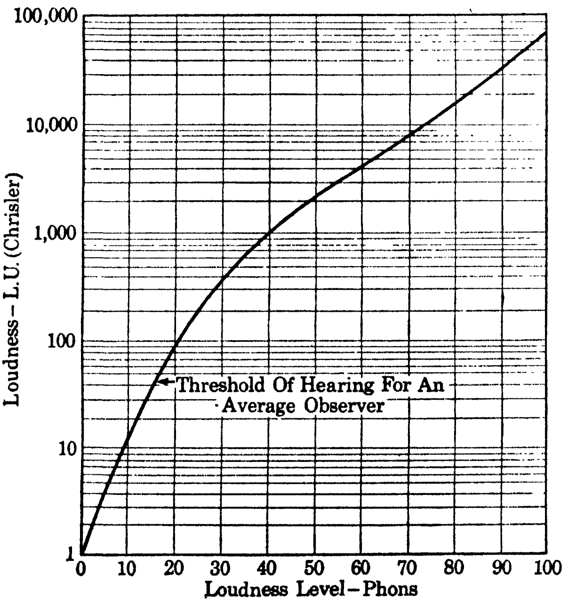

100,000
10,000
1,000
100
10
1
Loudness - L.U. (Chrisler)
Threshold Of Hearing For An Average Observer
0
10
20
30
40
50
60
70
80
90
100
Loudness Level - Phons

FIG. 8·3.

Attenuation is the reduction or difference in intensity expressed in decibels. As such the attenuation is given by the equation

$$\text{Attenuation} = 10 \log_{10} \frac{I_1}{I_2} = \text{db}_1 - \text{db}_2 \qquad [8 \cdot 5]$$

where I_1 and I_2 are the intensity values.

The *ambient level* is the level experienced under average conditions in any space and is analogous to ambient temperature.

Illustrative Problems.

1. Two fans have noise levels of 58 and 60 db, respectively. What is the noise level when they run simultaneously?

Solution. A noise level of 58 db is equivalent to an intensity of

$$I_1 = 6.31(10^5)I_0$$

and 60 db is equivalent to $I_2 = 1.0(10^6)I_0$. The combined intensity is $I_1 + I_2 = 1.631(10^6)I_0$ or

$$\text{db} = 10 \log_{10} \frac{1.631(10^6)I_0}{I_0} = 62.1$$

2. A room with an engine in it has a noise level of 90 db. An adjoining room has a level of 63 db. What is the attenuation through the wall.

Solution. The attenuation is the difference in levels, and so the answer is 27 db.

8·3. Character of Sound. The complex nature of sound makes difficult an accurate study of sound levels and their effect on people.[39] Sound being radiated from a surface is usually not susceptible to mathematical analysis. By using various simplifications, however, it is possible to explain many sound problems. It will be found that a sound that has a long wave length (low frequency) will radiate uniformly in all directions if the dimensions of the source are small compared to the wave length. If the wave length is small compared with the dimensions of the source, directive sound will result. This means that an average of several readings should be taken to establish the sound level if the sound source has any appreciable size. Figure 8·4 shows the relation between frequency and wave length for average conditions where the velocity of sound is 1120 ft/sec. It will also be found that high frequency sounds may pass through several maximum and minimum intensities before the intensity decreases as the square of the distance. Once it has reached these conditions, the sound intensity will decrease 6 db each time the distance from the source is doubled.

The test space also affects the noise level measured. Particular care should be exercised in testing equipment to make sure that vibration is

not carried to room surfaces. This is necessary to avoid any chance of sound production in the walls, ceiling, and floor. Sound will radiate from the source outward to the walls, ceiling, or floor and be reflected.[40,41] Therefore, any measurement of sound in a room will give the sum of the direct and the reflected sound. As mentioned before, the result of such a mixture is that sound intensity varies about the room in a complex manner. That is, the noise level will not vary as the square of the dis-

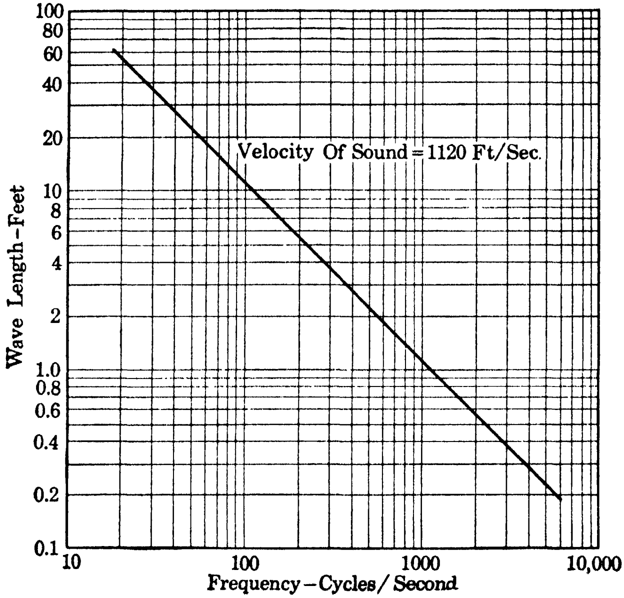


FIG. 8·4.

tance and may actually be louder at some points than at other points located closer to the source. For this reason it is necessary to investigate the noise throughout the room. In all cases the microphone and the unit under test should be located in the same positions in the same room for all tests which are to be compared. Therefore, the room, the location of the unit tested within the room, and the microphone location should be carefully noted in order that sound distributions will be similar.

In general, direct sound will predominate near the unit and give an indication of the sound-producing characteristics of the unit. Conversely, near the wall the reflected noise reaches its maximum and gives a poor indication of the unit's noise. A wide open space in which the sound waves would never be reflected would give a reading of the direct

noise. Small rooms, therefore, have the disadvantage of producing complex sound contours involving considerable reflected sound, such as shown by two examples in Fig. 8·5. The results would then have to be very critically examined to assure reasonable accuracy.

The room surface and room contents may also have a disturbing influence on sound measurements. The subject of room absorption has caused considerable discussion because it influences the noise level. Sound in essence is a form of energy which is dissipated by transmission to the outside or by absorption within the room. The sound intensity

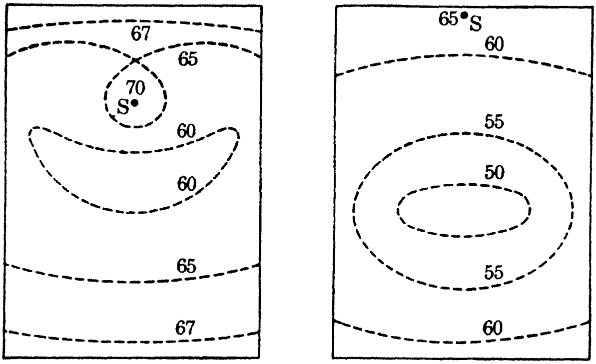


FIG. 8·5.

within the room will build up to a point where the sound energy lost will just equal the sound energy produced. This is analogous to problems in heat energy balance. For sound produced within the room, the transmission is negligible; therefore, in a room where there is no sound absorption or air damping, there would be an almost indefinite increase in sound level as long as sound energy production continues. The level within the room would also be uniform. If there were total absorption on the room surfaces, the sound would diminish as the square of the distance from the source. This is a condition reached only in wide open spaces where no reflective surface exists.

What is needed in a sound test room? If a highly reflective room is used, the measured level is too high, but the sound is very uniform so that location is not critical. Using a very dead room gives a truer reading of direct noise, but the noise varies considerably within the room and becomes directional. Room resonance conditions may also distort the relative importance of certain frequencies. Generally, the deadened room seems to be preferred.

To allow for different room conditions, a reverberation theory has been worked out which considers the average density in a room with

diffused reflected sound as inversely proportional to the total absorption in the room. From this it may be said that the difference in sound levels in two rooms varies as the logarithm of the ratio of the sabins of absorption in the rooms or

$$\text{db}_1 - \text{db}_2 = 10 \log_{10} \frac{A_2}{A_1} \qquad [8\cdot 6]$$

where $A_1 = \Sigma S_1\alpha_1$ sabins of room 1
$= \text{sum of (area} \times \text{absorption coefficient)}$
$A_2 = \Sigma S_2\alpha_2$ sabins of room 2
where S = area of surface in square feet
α = absorption coefficient from Table 8·1

Coefficients of absorption must be selected on the basis of the predominant frequencies.

A more accurate equation for use on dead rooms is

$$\text{db}_1 - \text{db}_2 = 10 \log_{10} \frac{S_2 \log (1 - \alpha_2)}{S_1 \log (1 - \alpha_1)}$$

The principal difficulty in using these equations is in determining the absorption coefficient. It is extremely difficult to measure the coefficient in an actual installation even where there is a reasonable amount of absorption. In making calculations it is best to use absorption coefficients from tables for acoustic properties of building materials found in architectural sound references or in Table 8·3.

8·4. Instruments. Instruments now commonly used for industrial noise measurement follow the American Standards Association's Tentative Standards Z24.3.[42, 43, 44] Therefore, all makes of instruments read the same levels within certain allowable limits. These small variations are due principally to the inability to make microphones which respond evenly throughout the frequency range. For this reason instruments of the same make may vary as much as instruments of different makes. In the range of most noise measurements, the variation is seldom over 1 or 2 db.

The ear has one characteristic which no instrument is able to measure, and that is judgment of noise quality. Rattles, squeaks, and irregular noises may not add enough sound energy to the total energy to be noticeable on the sound level reading, but they may be unpleasant to the ear. Therefore, to be complete, it is necessary that quality observations be made along with the actual sound level measurements.

8·5. Noise Measurement. When simple design changes in machinery are to be compared, one reading may be sufficient. When a more reliable measure of noise is desired, a series of readings around the unit

may be needed to check directional effects. Finally, when more general acceptance is established, it may be desirable to use the "sound power output" method so that all readings are reduced to a common datum point which is independent of distance and room conditions. This method is described later.

In measuring sound at one or more points it should be born in mind that results have meaning only when compared with other readings made under identical surroundings and location. The flat scale on sound-measuring instruments gives a measure of energy and can be worked with quantitatively. The weighted scale gives a measure of the ear response to the sound. The National Association of Fan Manufacturers recommends a distance of 1 fan diameter or not less than 5 ft at 7 definite points around the fan.[45,46] The actual distance used in any general problem depends principally on the room size and wall condition. Large distances may be used in large rooms or relatively dead rooms because the reflected sound will be small. Shorter distances are generally used in small rooms where reflected noise is to be minimized. The American Institute of Electrical Engineers recommends distances of 6 in., 1 ft, and 3 ft from the nearest major surface.[47]

Close measurements have the advantage of minimizing reflected sound, but, on the other hand, they emphasize the noise from closer portions of the unit being tested. Measurements at a distance tend to give the noises of different parts of a unit a true perspective even though they include considerable reflected sound and directional effect. Objections to close measurements can be largely overcome by taking a series of measurements around the unit. These can then be averaged on a simple arithmetic basis with sufficient accuracy if the readings are within a 12-db range. More accuracy can be obtained by converting decibel readings to power, averaging, and converting back to decibels. It is necessary that background noise be considerably less than the noise being measured. Table 8·1 shows what correction to apply when the background or ambient level is less than 10 db.

To summarize, any distance may be used for sound measurements if the readings taken are at least 10 db higher than the noise levels around the boundary walls. Also, the microphone should not be located near any wall or reflecting surface. Several readings around the unit give a more representative indication of the noise level than a single reading. The room noise level, without the unit operating, should be 10 db or more lower than the unit noise levels to eliminate errors in readings. If the difference in levels is less than 10 db, corrections should be made according to Table 8·1. Correction for room conditions can be made in an approximate manner by considering room absorption, as mentioned

TABLE 8·1

Db Difference between Total Noise and Background Noise	Db Error
1	6.9
2	4.3
3	3.0
4	2.2
5	1.7
6	1.3
7	1.0
8	0.8
9	0.6
10	0.5

previously, using equations 8·6 or 8·7. The absorption coefficients of the different areas in the room can be estimated from tables on acoustic properties and the total absorption determined. Further detailed information is given in the following discussion on a detailed sound level determination.

In the final analysis the quality of the noise must also be judged. Many times the sound level of one unit may be lower than some other unit, but the latter may actually sound better and cause less discomfort. Therefore, sound measurement not only is a matter of establishing or meeting some arbitrary number but also requires considerable judgment based on listening with the ear. Careful study of harmonics often explains what the ear considers objectionable.

8·6. Sound Power Output. A method of noise measurement which has gained favor recently is the sound power method.[48] As yet it has not been widely used. The chief object is to determine the sound power output from a unit. This means that an imaginary surface around the unit is divided into increment areas. By determining the level for each area, it is possible to evaluate the total sound power output from the unit within close limits. Since the sound would radiate outward, the area needed should be perpendicular to the sound travel. For any source of noise this would necessitate using a spherical area. With a unit on a hard floor, the noise would be reflected so that a hemisphere could be used. From a practical standpoint other surfaces could be used with little error. This method has the advantage that it is independent of microphone distance, and of test room surface condition and can be used for design purposes because the average noise level in any room may be predicted.

A detailed explanation of the sound power output method follows: The unit is located in a typical position with hard surfaces on the floor

or on the wall if it is normally near a wall. This assumes that sound energy radiated in this direction will be reflected. An imaginary surface is created around and reasonably close to the unit so that the unit is enclosed between the hard surfaces and the imaginary surface. The imaginary surface is divided into a number of small areas, the accuracy increasing with the number of areas.

The sound level on the flat scale is determined for each area. These readings are converted to power by calculation or use of suitable tables. As an example, 57 db corresponds to a power ratio of $5.012(10^5)$. The power in watts per square centimeter is then $5.012(10^5)(10^{-16})$. The power flowing through the increment area equals the power in watts per square centimeter times the area in square feet times 929 cm^2/ft^2. The total of the power through increment areas is the total sound output in watts.

If we consider the sound energy absorbed equal to the sound energy output, we can evaluate the average sound intensity in the room from the equation

$$I = \frac{E}{929\Sigma S\alpha} = \frac{\text{Power output}}{\text{Absorption}} \qquad [8\cdot 7a]$$

or the sound level will be given by

$$\text{db} = 10 \log_{10} \frac{I}{I_0} \qquad [8\cdot 7b]$$

where db = decibel reading on the flat scale
E = sound power output in watts
$\Sigma S\alpha$ = absorption in the room in sabins
$I_0 = 10^{-16}$ watts/cm^2

The absorption must be estimated from tables on properties of acoustical materials. Then, if the products of absorption areas and their coefficients are summed up, the average intensity may be determined by equations $8\cdot 7a$ and $8\cdot 7b$.

Several sources of error may contribute to inaccurate results. First is a neglect of ambient or background level. The reflected noise enters into the same consideration. If the noise level at the walls and ceiling are at least 10 db less than at the point of measurement, the original levels need not be corrected to give true output reading. Large equipment in a small room will cause the most trouble. For the average-shaped room, use of an average of wall readings is close enough in making a correction.

A major source of error is in estimating the absorption coefficient. It

is necessary to make a frequency analysis to determine the predominant frequencies. The absorption coefficient must be selected for the average predominant frequency. For rooms with considerable absorption, fairly accurate results may be obtained but, where low absorption exists, the errors become larger. An error of 25% in the absorption coefficient will give an error of one decibel.

Illustrative Problem. A small motor and fan were tested in a sound test room and in a low absorption room to determine the expected sound levels. The unit was placed so that it was reasonably free of interfering surfaces. Then an imaginary spherical surface was taken on a one-foot radius. One reading was taken on top, bottom, in front, in back, and on the two sides for a total of six readings which were 73, 75, 73.5, 78, 70, and 70. Each was considered to be representative of one-sixth the area of the sphere. From this the power output was determined to be 28.6 microwatts. No correction was considered worth while for wall-reflected noises, as the average reading 18 in. from the wall was 64.5 db compared with the average unit reading of 73.1 db. The ambient room level was about 50 db, and so no correction was needed.

Since the major disturbing frequencies are 30 to 200 cycles/sec, the absorption for the room was based on 128 cycles/sec. Rock wool of the type used has an absorption coefficient of 0.22. When 0.1 was used for the floor coefficient of absorption, the total absorption is 211 sabins.

The sound level as calculated was

$$\text{db} = 10 \log_{10} \frac{28.6(10^{-6})}{10^{-16}(929)211} = 61.64 \text{ db} \qquad [8 \cdot 8]$$

Measurements were made in the center of approximately 3-ft squares throughout the room. Since readings in the immediate vicinity of the fan are not indicative of general room level, they were eliminated. The average of the remaining readings was 63.8 db, giving a difference of about -2.2 db. This probably could have been improved if readings had been more carefully taken or the absorption better known.

To put the method to a further test, the unit was moved to a room with very little absorption. First, the absorption coefficient was estimated to be 0.1 for all surfaces. The absorption was calculated to be 76.7 sabins, and the calculated sound level was 66.0 db. When the actual sound readings were taken and averaged, the result was 67.0 db or a difference of -1 db. Considering how hard it is to obtain an accurate absorption coefficient, this is surprisingly close agreement.

As a matter of comparison the expected sound level was calculated by use of equation 8·6.

$$\text{db}_1 - \text{db}_2 = 10 \log_{10} \frac{A_1}{A_2} = 10 \log_{10} \frac{76.7}{211} = 4.4 \qquad [8 \cdot 9]$$

With the sound level in the sound test room 63.8 db, the sound level in the second test room should be $63.8 + 4.4 = 68.2$. The error here is about $+1.2$ db.

8·7. Frequency Analysis. To determine the causes of noise it is useful to know the predominant frequencies as well as their relative amplitude.

A frequency analyzer or band filter is used for this purpose. These instruments depend upon tuning, so that the amplitude of response for the different harmonics will be indicated. These harmonics are usually some multiple of the electric power frequency, the speed of the rotating parts, or the speed of some other moving part, panel, and so on. In some instances no distinct peaks are noted because the sound is so complex. This is usually the type of sound produced by shock effect and fluid flow. These are by far the hardest to trace and remedy.

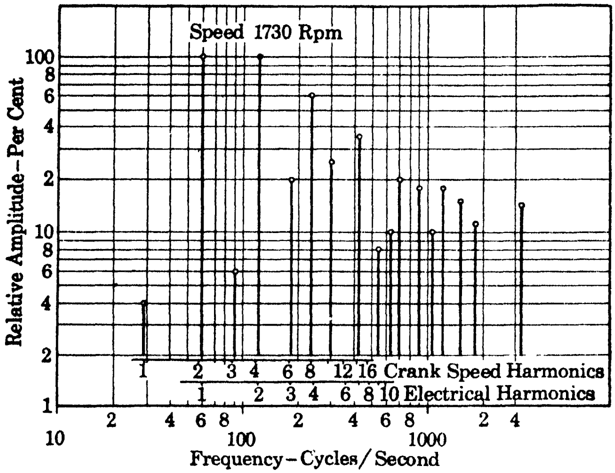


FIG. 8·6.

It is often best to record all frequencies and levels because it can then be seen whether peaks are sharp and clearly defined or if they are broad and show indications of many frequencies close together. When plotted either log-log or semi-log paper may be used. When a sufficient number of frequencies are plotted, the points may be connected to show the character of the frequencies, but if peaks only are plotted a line is drawn between the point and the X axis. The latter method is basically more correct and is illustrated in Fig. 8·6.

Either the weighted or flat networks may be used on the sound meter. The choice depends upon whether the effect of the different frequencies upon the ear is desired or the amount contributed by each frequency to the total sound energy.

8·8. Noise Sources. Noise originates in various manners. Unbalanced and shaking forces set up sympathetic vibrations in different members. Excessive clearances and impact may produce local vibrations of audible frequencies. Turbulent or pulsating fluid flow often is the most serious source of noise. Electric forces and vibrations can be very troublesome.

Each industry and type of product has its particular problems. A few examples will be considered together with an indication of some of the conflicting solutions to problems on the same equipment. In engines and compressors considerable noise is produced at the intake and exhaust or discharge. Gas passages are designed to eliminate sharp corners and edges and to provide constant velocities so that excessive turbulence is avoided. Mufflers reduce the noise to a more acceptable level. It is necessary to design mufflers to eliminate the most disturbing portion of the noise and still not offer much pressure drop through them. Airplanes have never adopted mufflers because of added weight and loss of performance. Ventilating fans and propellers produce most of their noise by turbulent flow and depend upon tip or peripheral speed. Fans must move a certain volume of air against a certain resistance. Large diameters running at slow speeds are the most quiet, but space and cost often oppose their use.

Clearances in bearings and between moving parts are often the source of noise. Reducing clearances reduces the noise but may give lubrication troubles. Panels act as sounding boards. One solution is to eliminate the panel or lighten it whereas the other is to stiffen it by increasing its weight or by adding stiffening bars and angles. Electric equipment is subject to magnetic forces, particularly 60 cycle and its harmonics. Mechanical equipment must be designed to avoid these disturbing frequencies.

8·9. Soundproofing. Since the noise at the source often cannot be economically reduced below the objectionable range, soundproofing is necessary to reduce the noise level to a comfortable value. Levels of less than 85 db are needed in order to carry on conversation. Soundproofing is accomplished by two methods: absorption of the sounds and reduction of the transmission of sounds through walls. The amount of sound energy transmitted through a wall depends upon the area of the wall and the characteristics of the wall, which are measured by the transmissivity coefficient τ. Expressed analytically this is

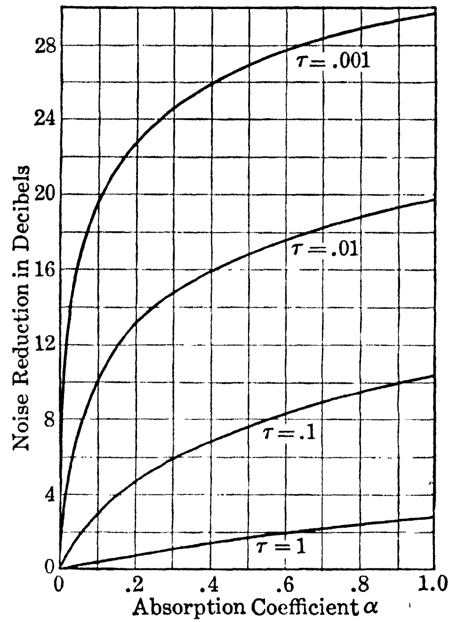
$$E_t = E_o \tau S \qquad [8\cdot10]$$

where E_t = energy transmitted through wall
E_o = energy on outside of wall
S = wall area in square feet
τ = transmissivity

The amount of sound energy absorbed on the inside when a sound strikes a wall also depends upon the area and a material factor known as the absorption coefficient. It is expressed as

$$E_a = E_i \alpha S \qquad [8\cdot11]$$

where E_a = energy absorbed by wall
E_i = energy striking wall
S = area
α = absorption coefficient



Courtesy N.A.C.A., Tech. Note 748

FIG. 8·7.

Since energy must be conserved, the sound energy transmitted through the wall will be equal to that transmitted back through the wall and that absorbed. This may be expressed analytically as

$$E_oS\tau = E_iS\tau + E_iS\alpha \qquad [8\cdot12]$$

This equation gives a means of finding the sound-reduction factor, or decibels lost in the wall, which is equal to

$$\text{db} = 10 \log_{10} \frac{E_o}{E_i} = 10 \log \frac{\tau + \alpha}{\tau} \qquad [8\cdot13]$$

Equation 8·13 considers the walls, ceiling, and floor as having the same coefficients, and the furnishings, accessories, and people inside as having no sound-absorbing properties. By considering the effect of each different area or body having soundproofing properties, the accuracy of the

calculations may be improved. We can evaluate the absorption and transmittance by the equations

$$\text{Absorption} = S\alpha = S_1\alpha_1 + S_2\alpha_2 + \cdots$$

$$\text{Transmittance} = S'\tau = S'_1\tau_1 + S'_2\tau_2 + \cdots \qquad [8\cdot 14]$$

The sound-reduction factor then becomes

$$\text{db} = 10 \log_{10} \frac{S\alpha + S'\tau}{S'\tau} = 10 \log_{10} \frac{S\alpha}{S'\tau} \qquad [8\cdot 15]$$

since $S'\tau$ is ordinarily very small compared to $S\alpha$. The effectiveness of different materials and combinations of soundproofing can be quickly judged by referring to Fig. 8·7.

8·10. Sound Transmission. Sound may be transmitted through walls in three ways: (1) It may go directly through holes or cracks; (2) it may vibrate the wall and set up a sound on the other side; (3) it may periodically compress the material of a rigid wall and be carried through the wall. It is very important that all holes or cracks be closed, for a crack only a few hundredths of an inch wide permits considerable noise to reach the inside as the value of $\tau = 1$. Considerable work has been done on walls of different types to find their sound-insulating values. Table 8·2 lists a few of the more commonly used values.

TABLE 8·2

TRANSMISSION LOSS

	Transmission Loss, db
Frame wall 2 × 4-in. studs 16-in. O.C. ½-in. Celotex lath, ½-in. plaster both sides	37
Same with staggered studs on 2 × 6-in. plate	42
Same with 2 × 2-in. studs on 2 × 6-in. plate ½-in. Celotex board between	52
Frame wall 2 × 4-in. studs 16-in. O.C. wood or metal lath ½-in. plaster both sides	33
4-in. hollow clay tile, ½-in. plaster both sides	40
6-in. concrete wall	42
8-in. brick, ½-in. plaster both sides	54
20-gage steel sheet	25
¼-in. steel door	35
1¾-in. solid oak door	25
1¾-in. fabricated door, usual hang	24
3-in. soundproof door	43
¼-in. glass plate	32
¼-in. glass plate double, 7-in. space	45
Wrapping paper	1.9
⅛-in. plywood	19
¼-in. plywood	21
Double strength glass	28

When walls are thin and light, it is easy for the sound source to vibrate the wall. This vibration can be prevented by putting stiffening ribs on the panels. If these additional ribs are symmetrically located they will be ineffective, as the mode of vibration is usually symmetrical. Pieces of insulating material cemented to the inner side of the panels also reduce the sound transmission very effectively. Tests on sample panels by the National Bureau of Standards show that the mass of the insulation is the most important factor in determining the transmission through a panel. When double walls or panels are used, the transmission is lower particularly when the two are isolated from each other. In tables, the transmissivity is often given. This may be converted to decibel loss by the equation

$$\text{Transmission loss in decibels} = 10 \log_{10} \frac{1}{\tau} \qquad [8\cdot16]$$

8·11. Sound Absorption. Sound in an inclosure must be dissipated by absorption. Absorption is promoted in two ways, by porosity and by diaphragm action. If the material is porous the sound energy is absorbed by friction as it strikes the surface and penetrates. The remainder is reflected. Thick layers naturally absorb more as the sound waves have farther to travel before they are reflected off the back surface. The absorption is approximately proportional to the thickness at low frequencies. In diaphragm action, energy is used up in making the diaphragm vibrate. Therefore, it makes a great deal of difference how the material is fastened to the wall. Cementing it to a panel prevents diaphragm action and so reduces the absorption at low frequencies but makes little difference at high frequencies. The tension in the material also affects the absorption. It has been shown that the single diaphragm behaves as a single mass system wherein the diaphragm is the mass and the air space acts as the spring when it is set away from the wall. Therefore, the absorption increases up to the resonance frequency and then drops. When several layers are used they act as a filter in electric circuits and pass the higher frequencies more readily than the lower ones. Table 8·3 gives the absorption coefficients for some of the more common materials. The materials generally used in building soundproofing are most effective at higher frequencies. They are usually hard materials which tend to reflect low frequency sounds. Airplane soundproofing requires softer materials to eliminate the low frequencies that are the major source of noise. The problem in planes is also complicated by the fact that the available area for absorption in cabins is insufficient.

Architectural sound-absorbing materials are used to improve the hearing conditions and to reduce the noise level. They are often per-

TABLE 8·3

SOUND ABSORPTION

Material	128	256	512	1024	2048	lb/ft^2
	Absorption Coefficient, cycles per second					
Brick wall, unpainted	0.024		0.03		0.049	
Painted	0.012		0.017		0.023	
Plaster on lath, smooth	0.02		0.03		0.04	
On tile or brick	0.013		0.025		0.04	
Acoustic plaster	0.25		0.40		0.50	
Wood paneling	0.08		0.06		0.06	
Glass	0.035		0.027		0.02	
Marble and glazed tile	0.01		0.01		0.015	
Floors, concrete or terrazzo	0.01		0.015		0.02	
Wood	0.05		0.03		0.03	
Linoleum, etc., on concrete	0.03		0.03		0.04	
Carpet, no base	0.09		0.20		0.27	
Felt base	0.11		0.37		0.27	
Fabrics hung						
Light 10 oz 1 yd^2	0.04		0.11		0.30	
Medium 14 oz 1 yd^2	0.06		0.13		0.40	
Heavy 18 oz 1 yd^2	0.10		0.50		0.82	
Acoustic tile, average	0.30		0.45		0.60	
Rock Wool Blanket, 1″	0.29	0.72	0.83	0.78	0.72	1.2
2″	0.66	0.74	0.84	0.88	0.83	2.4
Air acoustic, $\frac{1}{2}$″	0.24	0.46	0.50	0.57	0.63	0.75
1″	0.46	0.60	0.63	0.66	0.73	1.5
Hair felt, 1″	0.06	0.27	0.57	0.77	0.81	0.7
Acousti-Celotex (C-1) $\frac{1}{2}$″	0.07	0.14	0.57	0.59	0.64	0.76
(C-4) $1\frac{1}{4}$″	0.28	0.56	0.98	0.78	0.59	1.48
Absorbex (*A*), 1″	0.41	0.71	0.96	0.88	0.85	2.21
Q-T Ductliner, $\frac{1}{2}$″	0.14	0.38	0.43	0.76	0.75	0.57
Dry Zero in muslin, 2″	0.28	0.48	0.68	0.82	0.98	0.50
Glass Wool, 1″	0.20	0.66	0.92	0.93	0.83	0.34
CJ Seapak, $\frac{1}{2}$″	0.19	0.23	0.99	0.62	0.61	0.22
People, in hard seats	1.0 sabin		3.0 sabins		3.5 sabins	
In upholstered seats	2.0 sabins		4.3 sabins		6.0 sabins	
Chairs, hard	0.15 sabin		0.17 sabin		0.20 sabin	
Upholstered			3.0 sabins			

forated to increase the surface exposed to the incident sound. Covering a sound absorbing material with a perforated metal sheet has little effect on the sound-absorption properties since sound is able to pass around the obstacles as long as the obstacles are small compared with the sound wave length. When sound-absorbing material is painted, the pores are closed and the absorption properties are lost except when the

perforated holes are used. Then there is little effect lost as the sound is able to reach the material and be absorbed.

Sound-absorbing material has many forms. Acoustic plaster is made by forming minute gas bubbles to give porosity. Some add mineral fiber for greater absorption. Blanket types of sound absorption are usually cemented or hung loosely on a surface. Most installations of this type must be protected by a loosely woven material or perforated metal sheet. Sound-absorption boards are available in various sizes. They are composed of different fibers held together by some binding agent. Tiles are available for more decorative treatments. These are often produced with perforation holes to improve the absorption.

In selecting sound-absorption materials several factors must be considered among which are:

1. Soundproofing qualities.
2. Heat-insulation value.
3. Light-reflection value.
4. Appearance.
5. Weight.
6. Adaptability.
7. Flame resistance.
8. Vermin resistance.
9. Non hygroscopicity.
10. Cost.

8·12. Design of Soundproofing. When sound is to be kept out it is necessary to design a wall or panel such that the attenuation reduces the noise level to an acceptable value. It is very necessary to eliminate holes and cracks through which the sound might pass. The final value also depends upon the absorption present and can be calculated from equation 8·15. When holes for ventilation or duct work are necessary, the ducts must be lined with sound-absorbing material to reduce the level.

The sound insulation needed depends upon the amount of noise in the room under consideration. If a sound level of 70 db is present on one side of the wall and there is a 40-db loss in the wall, the sound level getting through will be 30 db. If the room is otherwise quiet, the 30-db sound will be heard. If the noise level would be 30 db anyway, the sounds could still be heard, but, if the normal level in this room is 40 db or above, the noise from the other room could not be heard. This is known as masking. To mask conversation it is usually necessary to have a transmission loss of over 30 db. Over 45 db is needed for very loud sounds such as from singing and musical instruments.

When the walls have doors, windows, or openings in them the transmission loss can be found from the equation

$$\text{db} = 10 \log_{10} \frac{\Sigma S}{\Sigma S\tau} \qquad [8\cdot 17]$$

where S = area in square feet
τ = transmission coefficient

Illustrative Problem. A wall 10 × 20 ft has a door located in it. The wall has a transmission loss of 35 db while the door has a transmission loss of 25 db. The door is 3 × 7 ft. What is the expected transmission loss?

Solution. The transmissivities are

$$35 = 10 \log_{10} \frac{1}{\tau_1} \qquad \tau_1 = 0.000316$$

$$25 = 10 \log_{10} \frac{1}{\tau_2} \qquad \tau_2 = 0.00316$$

The loss is therefore

$$\text{db} = 10 \log_{10} \frac{200}{21(0.00316) + 179(0.000316)} = 32.1 \text{ db}$$

The design of the absorption depends on its application and purpose. If a room has insufficient absorption the sound persists and conversation and music become confusing. The reverberation time is a measure of the acoustics in a room. If the reverberation time is too long, echoes will be present, but, if it is too short, the sound may not be sufficiently loud in all portions of the room. The reverberation time can be calculated from the equation

$$T = \frac{0.05V}{S_a} \qquad [8\cdot 18]$$

where T = reverberation time in seconds
V = room volume in cubic feet
S_a = absorption in sabins

The optimum reverberation time depends on the size of the room and can be obtained from Fig. 8·8. The lower limit is most suitable for speech, the upper limit is most suitable for music, and the middle is best for general purpose use. If a room has insufficient absorption the deficit must be made up with sound-absorbing material.

Illustrative Problem. A room whose volume is 15,400 cu ft has a sound absorption of 400 sabins. What is the reverberation time and what is the approximate sound absorption needed in a room this size?

Solution. The reverberation time is

$$T = \frac{0.05(15{,}400)}{400} = 1.63 \text{ sec}$$

The optimum time is approximately 1.02 sec. This requires an absorption equal to

$$S_a = \frac{0.05(15{,}400)}{1.02} = 755 \text{ sabins}$$

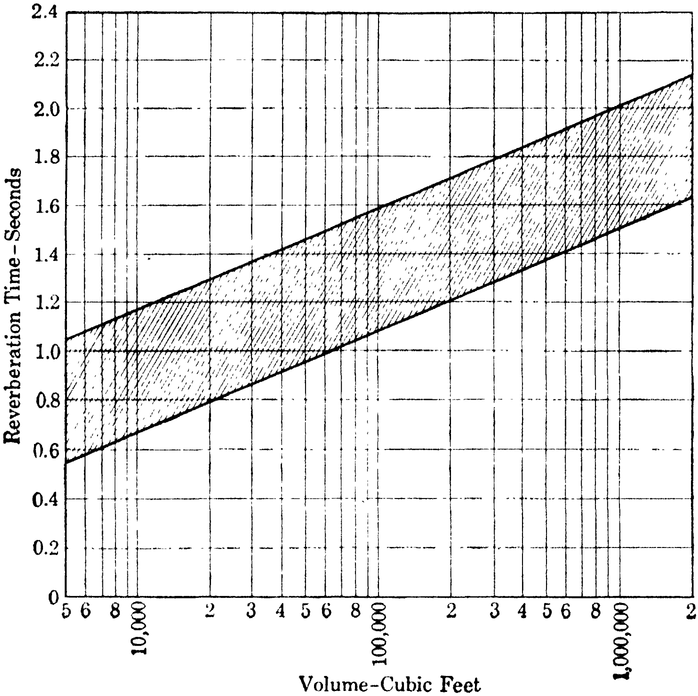


FIG. 8·8.

When quieting is needed considerably more absorption is used. The sound level is dependent on the absorption and can be calculated according to equation 8·6. It should be pointed out that the coefficient now generally used in calculating the absorption is the average of the coefficients at the frequencies of 256, 512, 1024, and 2048 cycles/sec.

Illustrative Problem. A room with a noise level of 70 db has 150 sabins absorption. Sound absorption with an average factor of about 0.72 over the range of predominant frequencies is now applied on the ceiling which is 20 × 50 ft. What is the expected sound level reduction? What is the reduction of the loudness?

Solution. The original plaster had a factor of about 0.03, and so the gain will be equal to 0.69. The absorption added will be 20(50)0.69 = 690 sabins. The total absorption will be 690 + 150 = 840 sabins. The sound reduction will be

$$\mathrm{db}_1 - \mathrm{db}_2 = 10 \log_{10} \tfrac{840}{150} = 7.5$$

A change in level from 70 db to 62.5 db reduces the loudness from 8000 to 5000 or 37.5% reduction.

Usually a reduction of 5 or 10 db is considered satisfactory to most offices. Noisy machines are usually grouped together and the walls and ceiling are covered with sound-absorbing material. Partial walls of sound-absorbing material are often added to shield other portions of the room from the direct sound. Similar methods are used in ventilating ducts. Absorption material is added to duct walls, plenum chambers are added, baffles are used to eliminate direct sound, and plate absorbers are placed in the air path to absorb sound.

PROBLEMS

8·1. An air-conditioning unit operates with a noise level of 65 db. When it is used in a room with an ambient level of 50 db, what is the resultant noise level? If the room level is 60 db? 65 db? 70 db?

8·2. One lathe operates at a level of 68 db, and another one operates at 73 db. What is the noise level when both are operating?

8·3. Five calculating machines each have a noise level of 70 db. What is the noise level when 2, 3, 4, and 5 machines operate simultaneously?

8·4. A compressor is tested by the sound output power method. It is set on a hard floor so that sound is reflected. At a radius of 6 ft five readings are taken near the center of five equal areas on a spherical surface. These readings are 78, 80, 79, 82, and 83. What is the sound output power in watts? What is the expected level when the compressor is installed in a room with 86 sabins absorption? With 600 sabins absorption?

8·5. A room air conditioner has a noise level of 65 db in a room with 150 sabins absorption. What is the expected level when the ceiling is treated with sound-absorbing material which brings the absorption to 930 sabins?

8·6. An office has a level of 70 db which is considered too noisy, and so sound-absorbing material is added to the ceiling. Originally the room has an absorption of 98 sabins but after the absorption is treated it is 1020 sabins. What is the resultant noise level expected?

8·7. A general office has a noise level 68 db without any sound-absorbing material. The ceiling which is 20 × 30 ft was originally bare plaster on lath. It is treated with Absorbex (*A*). If the initial absorption was 145 sabins, what will be the final value of room noise level?

8·8. An auditorium with a volume of 100,000 ft³ is to be designed for optimum soundproofing. What is the optimum amount of sound absorption needed?

8·9. A lecture room 20 × 50 × 10 ft now has an absorption value of 150 sabins

when empty. It is hoped to have optimum sound conditions when the room is has full, that is when the audience numbers 80. If each person adds 3.0 sabins what if the amount of absorption needed? What is the reverberation time when the room is full?

8·10. A 6-in. concrete wall 10 × 20 ft with a transmission loss of 42 db has a 6 × 6-in. hole for a small pipe to pass through. What is the transmission loss after the hole is made?

8·11. An 8 × 20-ft wall with a transmission loss of 37 db has a 30 × 76-in. door mounted in it. The door is a $1\frac{3}{4}$-in. fabricated door. What is the expected transmission loss?

8·12. An airplane is soundproofed according to the following schedule. What sound reduction can be expected?

Item	Area, sq ft	α
Ceiling	400	0.85
Front wall	53	0.88
Rear wall	62	0.88
Side walls	470	0.83
Floor	200	0.36
20 passengers at 3 sabins		
20 chains at 2.8 sabins		
Trim and equipment at 30 sabins		
		τ
Cabin ceiling and walls	985	0.00062
Floor	200	0.0012
10 windows, each 2.5 sq ft	25	0.0074
Door	18	0.0004

8·13. An office has a noise level of 70 db when the walls and ceiling are bare and have a total absorption of 89 sabins. How much absorption must be added to reduce the level to 60 db?

8·14. A laboratory with a noise level of 85 db now has 120 sabins absorption. How many square feet of Acousti-Celotex (C-4) $1\frac{1}{4}$-in. must be added to reduce the noise level to 70 db?

CHAPTER IX

THE MOBILITY METHOD

9·1. Introduction. Previous methods of solving systems having several masses both with and without damping have depended upon the use of differential equations. The solutions of the resulting differential equations for several masses are involved and become very difficult when damping is considered. One method which has been proposed for solving these vibration problems is the mobility method developed by Firestone.[49, 50, 51, 52] This method eliminates the use of differential equations. Relative and absolute displacements, velocities, accelerations, and forces may be found for any point in the system by using algebraic methods. Damping in differential equations makes the problems unwieldy, but the mobility method handles damping very easily.

The mobility method uses complex notation to represent displacements, velocities, accelerations, and forces. In this method displacements, velocities, and accelerations are considered individually for each member in the vibrating system. In this way the effects of all the members may be combined to find the motion of the system as a whole. A term, the mobility or ease of motion, is used to describe the behavior of each member. A schematic diagram called the mobility diagram is drawn to represent the vibrating system just as the electric circuit diagram represents an electric system.

9·2. Complex Notation. In the first chapter it was shown that a vibration could be represented by a rotating vector. This vector was described by sine and cosine terms. The mobility method uses complex numbers to represent vectors instead of trigonometric quantities. The resulting operations with the complex members are mainly algebraic. Complex numbers can be used to indicate the magnitude and position of a vector. The length of a vector can be expressed in complex notation by taking the projections on two perpendicular coordinates.

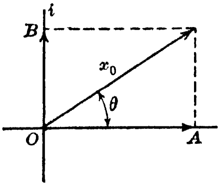


FIG. 9·1.

Thus the vector in Fig. 9·1 is represented by the notation $A + Bi$, where A represents the projection on the abscissa and B the projection on the ordinate. Later it will be shown that B is

multiplied by $i(\sqrt{-1})$ because it is 90° ahead of A, which is the real part of the number. In Fig. 9·1 we see that the vector may be represented in terms of the sine and cosine so that

$$A + Bi = \sqrt{A^2 + B^2}\,(\cos\theta + i\sin\theta) \qquad [9\cdot 1]$$

Either of these forms is cumbersome, particularly when they are differentiated. They are more conveniently expressed in polar form. It is shown in most advanced mathematics books that the trigonometric part of equation 9·1 can be expressed as

$$\cos\theta + i\sin\theta = e^{i\theta} \qquad [9\cdot 2]$$

Equation 9·1, therefore, can be written as

$$A + Bi = \sqrt{A^2 + B^2}\,e^{i\theta} = x_0 e^{i\theta} \qquad [9\cdot 3]$$

where

$$x_0 = \sqrt{A^2 + B^2} = \text{length of the vector} \qquad [9\cdot 4]$$

$$\theta = \tan^{-1}\frac{B}{A} \qquad [9\cdot 5]$$

where the signs of A and B must be retained for the proper location of the vector. Either form can be used to represent and locate a vector. The form to use depends on the type of calculation to be made. Changing from one to another is necessary at times.

The complex numbers in the polar form are very convenient for representing a rotating vector. This is particularly true when velocities and accelerations are to be obtained from displacements. This advantage can be recognized by considering the following general example wherein a displacement is represented as

$$x = x_0 e^{i\theta} \qquad [9\cdot 6]$$

because the angular position is dependent on the angular velocity ω, and the time t we can write

$$x = x_0 e^{i\omega t}$$

Either the real or the imaginary component could be used to represent the displacement at any point during a cycle.

The velocity, the first derivative of the displacement, is given as

$$v = \frac{dx}{dt} = ix_0\omega e^{i\omega t} = iv_0 e^{i\omega t} \qquad [9\cdot 7]$$

where $v_0 = x_0\omega$.

When a term is multiplied by i it indicates that the vector has been

rotated ahead 90°. This is proved by substituting $\theta = 90°$ in equation 9·2, which then becomes

$$e^{i90°} = \cos 90° + i \sin 90° = i$$

When a complex number is multiplied by i, the result is

$$ie^{i\theta} = e^{i90°}e^{i\theta} = e^{i(\theta+90°)} \qquad [9\cdot 8]$$

From this we see that the velocity is given by the equation

$$v = v_0 e^{i(\omega t+90°)} \qquad [9\cdot 9]$$

The velocity vector, therefore, leads the displacement vector by 90°.

The acceleration is the second derivative of the displacement so that

$$a = \frac{d^2x}{dt^2} = i^2 x_0 \omega^2 e^{i\omega t} = a_0 e^{i(\omega t+180°)} \qquad [9\cdot 10]$$

where $a_0 = x_0\omega^2$ = amplitude of the acceleration.

The acceleration is 180° ahead of the displacement vector located by ωt. These quantities and their complex form are listed in Table 9·1.

TABLE 9·1

LINEAR SYSTEMS

	Instantaneous Value	Amplitude
Displacement	$x = x_0 e^{i\omega t}$	x_0
Velocity	$v = \frac{dx}{dt} = v_0 e^{i\omega t}$	$v_0 = i\omega x_0$
Acceleration	$a = \frac{d^2x}{dt^2} = a_0 e^{i\omega t}$	$a_0 = i\omega v_0 = -\omega^2 x_0$
Force	$F = F_0 e^{i\omega t}$	F_0

TORSIONAL SYSTEMS

Angular displacement	$\theta = \theta_0 e^{i\omega t}$	θ_0
Angular velocity	$\Omega = \frac{d\theta}{dt} = \Omega_0 e^{i\omega t}$	$\Omega_0 = i\omega\theta_0$
Angular acceleration	$A = \frac{d\Omega}{dt} = A_0 e^{i\omega t}$	$A_0 = i\omega\Omega_0 = -\omega^2\theta_0$
Moment	$T = T_0 e^{i\omega t}$	T_0

In the application of complex notation to vibration problems addition, subtraction, multiplication, and division are used, making it necessary to resort to use of the rectangular form of these equations. For addition real terms are added to real terms and imaginary terms are added to imaginary terms. An example is

$$(A + Bi) + (C + Di) = (A + C) + (B + D)i \qquad [9\cdot 11]$$

Subtraction is done in the same manner. Multiplication is carried on by the usual algebraic methods. For instance

$$\begin{aligned}(A + Bi)(C + Di) &= AC + ADi + CBi + DBi^2 \\ &= (AC - BD) + (AD + CB)i \qquad [9\cdot 12]\end{aligned}$$

In division it is necessary to multiply both numerator and denominator by the conjugate complex of the denominator; that is,

$$\frac{A + Bi}{C + Di} = \frac{(A + Bi)(C - Di)}{(C + Di)(C - Di)} = \frac{(AC + BD) + i(BC - AD)}{C^2 + D^2} \qquad [9\cdot 13]$$

Both multiplication and division may be carried on using the polar form so that

$$x_1e^{i\theta_1}x_2e^{i\theta_2} = x_1x_2e^{i(\theta_1+\theta_2)} \qquad [9\cdot 14]$$

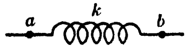
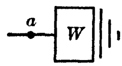
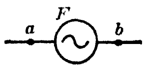
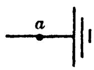
and

$$\frac{x_1e^{i\theta_1}}{x_2e^{i\theta_2}} = \frac{x_1}{x_2}e^{i(\theta_1-\theta_2)} \qquad [9\cdot 15]$$

9·3. Basic Elements and Definitions. In any vibrating system there are only three kinds of members, namely: springs or elasticities, masses or inertia, and damping devices or viscous friction. These members are represented schematically as shown in Table 9·2.

The following are several new terms and expressions which will be used in succeeding pages. *Terminals* are those points on a member in the vibrating system where forces are applied. Thus, in Table 9·2, points *a* and *b* represent terminals. A *spring* and a *damper* both have two terminals because two equal and opposite forces must be applied to maintain equilibrium. A *mass* has only one because the inertia force requires no physical terminal. To indicate that there is a *reactive* or *inertia force* depending on the absolute motion, the symbol for a fixed point or fixed terminal is added. Whenever one end of a spring or damper is fixed, the member is said to be *grounded*. Masses are always grounded, that is, have one fixed terminal, because the inertia force depends on the absolute motion of the mass. The terminals also determine the *displacements, velocities*, and *accelerations* of one end of a spring

TABLE 9·2

Item	*Symbol*
Spring or shaft	
Damper or friction	
Mass or disk	
Force or moment	
Fixed point	

or damper relative to the other end. These relative displacements, velocities, and accelerations of the terminals are said to be *across* the members. Since a force acting on one terminal of a spring or damper is balanced by an equal and opposite force on the other terminal, the force is said to act *through* the member.

When masses or individual points in the system are dealt with, only the absolute motion is of interest because it determines the *absolute force* on the mass or point. The inertia force on the mass requires no physical terminal. We say, therefore, that the force acts *on* the point or mass and that the point or mass has a displacement, velocity, and acceleration.

Analogous expressions are used for moments acting *through* shafts and dampers and for moments acting *on* points and inertia disks. *Relative angular motion* is said to occur *across* shafts and dampers.

Elastic members are assumed to follow Hooke's law so that the displacement of one terminal a of the spring relative to the other terminal b is proportional to the force applied to it. This statement is shortened by saying that the displacement *across* the spring is proportional to the force acting *through* it. Dampers such as dash pots resist movement with a force proportional to the velocity of one terminal a of the damper relative to the other terminal b. This may be restated to read: The velocity *across* a damper is proportional to the force acting *through* the damper. For the mass the statement is somewhat different. Newton's law states that the acceleration is proportional to the force applied to the mass. Here the acceleration is an absolute quantity measured rela-

tive to a fixed point. It is then said that the acceleration of a mass is proportional to the force acting *on* the mass.

9·4. Mobility. In the previous analyses based on differential equations, a force analysis is made which is essentially a measure of the difficulty or inability of the system to move because of the inertia force. The mobility method is in effect just the opposite for it gives a measure of the mobility or ease of motion. It is defined as the velocity amplitude divided by the force amplitude. For any member it is then equal to the velocity *across* the member divided by the force *through* the member, and may be expressed in equation form as

$$z = \frac{v_0 e^{i\theta_1}}{F_0 e^{i\theta_2}} = \frac{v_0}{F_0} e^{i(\theta_1 - \theta_2)} = \frac{v}{F} \qquad [9\cdot 16a]$$

or in the torsional system

$$z = \frac{\Omega_0 e^{i\theta_1}}{T_0 e^{i\theta_2}} = \frac{\Omega_0}{T_0} e^{i(\theta_1 - \theta_2)} = \frac{\Omega}{T} \qquad [9\cdot 16b]$$

The ease of motion is given by the absolute value of v_0/F_0 or Ω_0/T_0, whereas the angle represents the phase angle by which the velocity leads the force or moment vector. A negative value means that it lags. The mobility of any point in the system is likewise the ratio of the velocity amplitude of the point to the force amplitude, acting on the point.

Springs, dampers, and masses all require forces to produce velocities across their terminals. Each, therefore, has an effect upon the whole system. This total effect is easily determined by combining the effects of the individual members. These effects are most easily described by the individual mobility because each can be evaluated in terms of the physical characteristics and the frequency of vibration.

In springs the displacement amplitude in terms of the force and spring constant is

$$x = \frac{F}{k}$$

The velocity amplitude by differentiating is

$$v = i\omega x = i\omega \frac{F}{k}$$

When this value is substituted in equation 9·16*a*, the mobility becomes

$$z = \frac{v}{F} = \frac{i\omega F}{Fk} = \frac{i\omega}{k} \qquad [9\cdot 17]$$

This shows that weak springs represented by small k values are much easier to move than stiffer ones. It can be shown that all imaginary terms indicate that energy oscillates in and out of the member without loss. Therefore, any energy put into a pure spring can be recovered again.

For dampers, the fundamental assumption that velocity is proportional to the force gives the velocity as

$$v = \frac{F}{r}$$

so that the mobility is

$$z = \frac{v}{F} = \frac{F}{rF} = \frac{1}{r} \qquad [9 \cdot 18]$$

Large values of damping will naturally reduce the movability of the system. This mobility is a real number and as such it indicates that energy is dissipated when there is action. This loss cannot be recovered again.

For a mass the force is

$$F = ma = \frac{W}{g} a$$

where the acceleration from Table 9·1 is

$$a = i\omega v$$

so that the velocity becomes

$$v = \frac{a}{i\omega} = \frac{F}{i\omega m}$$

The mobility, therefore, is

$$z = \frac{v}{F} = \frac{F}{Fi\omega m} = \frac{1}{i\omega m} \qquad [9 \cdot 19]$$

Large masses give large inertia forces and lower mobility as would be expected. This expression too is imaginary so that it indicates that kinetic energy is alternately stored in the mass and then released.

For torsional systems an equivalent mobility may be evaluated for the disks, shafts, and dampers. The resulting terms are listed in Table 9·3 along with the linear values.

These mobilities when combined properly give the mobility of the whole system. Members in a vibrating system can be connected in one of two ways, series or parallel combinations. A force may be applied to one end of a spring which is attached at the other to a mass, as shown in

TABLE 9·3

MOBILITIES FOR INDIVIDUAL MEMBERS

Member	Mobility
Linear System	
Spring	$\dfrac{i\omega}{k}$
Damper	$\dfrac{1}{r}$
Mass	$\dfrac{1}{i\omega m} = \dfrac{g}{i\omega W}$
Torsional System	
Shaft	$\dfrac{i\omega}{k_t}$
Damper	$\dfrac{1}{r_t}$
Inertia disk	$\dfrac{1}{i\omega I}$

Fig. 9·2*a*. The force applied to the spring must pass through the spring to the mass. Thus we say that the members are in series. The velocity at the point *b* where the force is applied to the system is equal to the

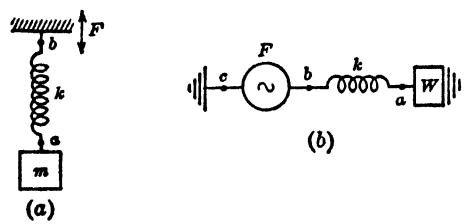


FIG. 9·2.

velocity of the mass plus the velocity change across the spring. The mobility of the terminal where the force is applied is

$$z = \frac{v_b}{F_b} = \frac{v_a + v_{ba}}{F_b} = \frac{v_a}{F_b} + \frac{v_{ba}}{F_b} = \frac{v_a}{F_a} + \frac{v_{ba}}{F_{ba}} = z_a + z_{ba}$$

This example shows that when members are acted on by the same force and when the total velocity across the members is equal to the sum

of the individual velocities, the members are in series and the mobility across the members is equal to the sum of the individual mobilities. This can be expressed by the general equation

$$z = z_1 + z_2 + \cdots z_n \qquad [9\cdot 20]$$

It is thus twice as easy to move two identical springs in series as it is to move one alone.

Another combination is shown in Fig. 9·3*a*, where the force is acting

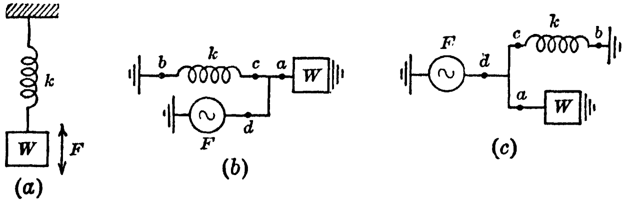


FIG. 9·3.

on the mass and the spring is fastened to the fixed frame. The forces at the mass as shown in Fig. 9·3*b* are the spring force, the inertia force of the mass, and the applied force. The applied force relative to the frame is equal to the sum of the other two. The velocity across the spring and the velocity of the mass are the same as the velocity of the point *d* in the system where the force acts. The mobility of a point on the mass where the force is applied is therefore

$$z = \frac{v_d}{F} = \frac{v_d}{F_a + F_{cb}} = \frac{1}{\dfrac{F_a}{v_d} + \dfrac{F_{cb}}{v_d}} = \frac{1}{\dfrac{F_a}{v_a} + \dfrac{F_{cb}}{v_{cb}}} = \frac{1}{\dfrac{1}{z_a} + \dfrac{1}{z_{cb}}}$$

This example shows that when the relative velocity is the same across several members and the force through the members is the sum of the individual forces, the members are in parallel and the mobility can be determined by the relation

$$z = \frac{1}{\dfrac{1}{z_1} + \dfrac{1}{z_2} + \cdots + \dfrac{1}{z_n}} \qquad [9\cdot 21]$$

This equation shows that it is half as easy to move two identical springs in parallel as it is to move one alone. The solution to systems of parallel members is facilitated by drawing the diagram in Fig. 9·3*b* in a little different form, wherein the applied force is opposed by all the

other forces, as shown in Fig. 9·3c. Here the parallel arrangement becomes more obvious.

9·5. Schematic Diagrams. Mechanical systems in vibration are much easier to analyze when a schematic diagram is drawn first. Very simple

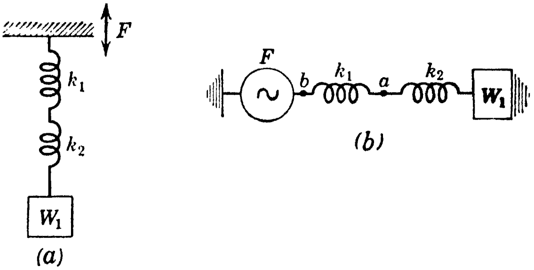


Fig. 9·4.

but fundamental diagrams were included in the section above. More complex systems may be drawn up using the same principles and rules. The basic rule is to have the applied force opposed to all other forces in the system. It is convenient to have the harmonic force fixed or grounded on the left side of the diagram. The other members depend-

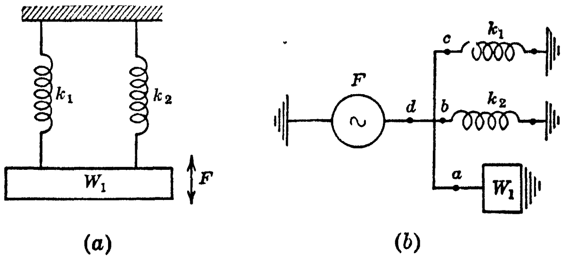


Fig. 9·5.

ing on absolute motion are grounded to the right. Each member must maintain the same position in the diagram that it has in the actual system. Therefore, in Fig. 9·3c the spring and the mass are both grounded to the right while the force is grounded to the left. Each member is very distinctly set off from every other member even when the terminals are really fastened together. This is necessary when combining mobilities. The easiest way to learn how to draw diagrams is to follow the examples shown in Figs. 9·4 to 9·12. Figure 9·9 represents a reduced equivalent system such as is discussed in Chapter VI.

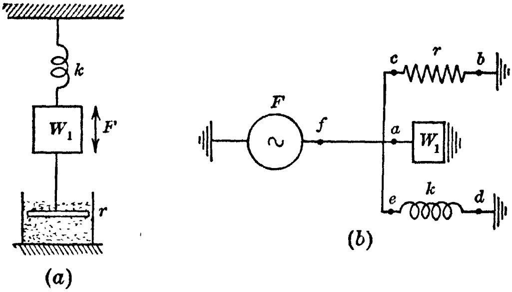


FIG. 9·6.

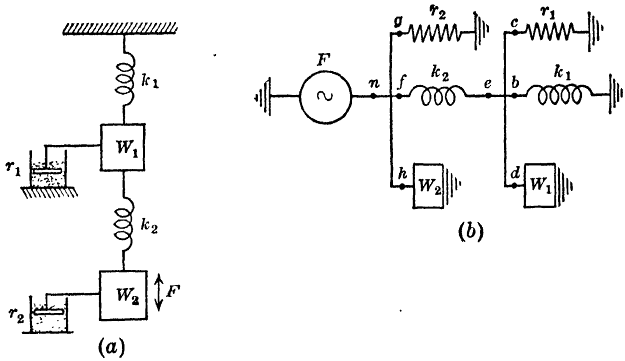


FIG. 9·7.

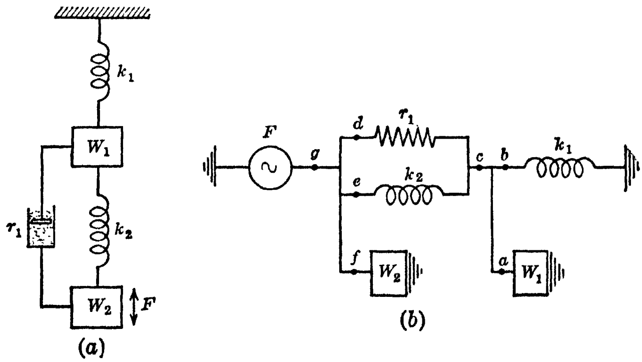


FIG. 9·8.

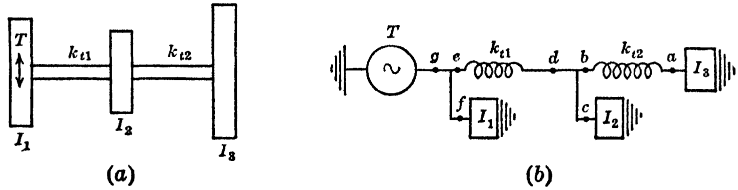


FIG. 9·9.

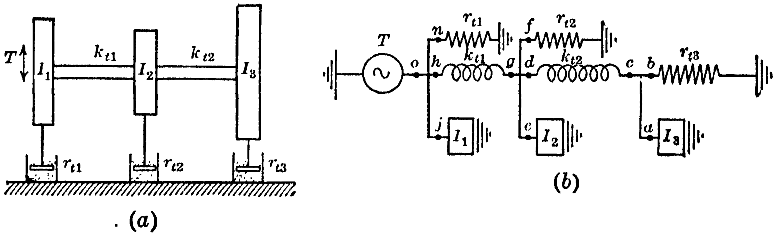


FIG. 9·10.

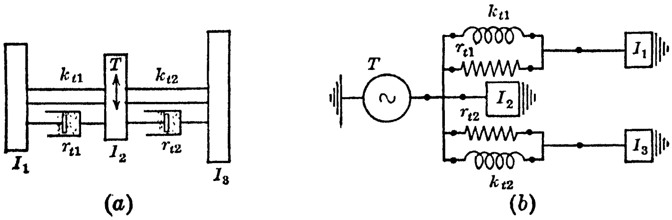


FIG. 9·11.

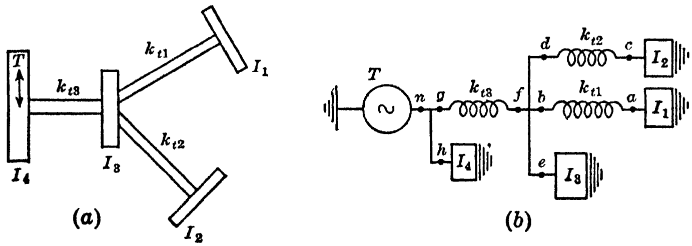


FIG. 9·12.

9·6. Calculation of Mobilities. When the schematic diagram is complete it becomes rather evident which members are in series and which are in parallel. The first rule in finding mobilities is to proceed from

the right end toward the impressed force. Thus from Fig. 9·3c we may write that the mobility at point a is

$$z_a = \frac{1}{i\omega m}$$

The mobility across cb is the same as the mobility of c, which is

$$z_c = \frac{i\omega}{k}$$

The mobility of d is determined from the two mobilities in parallel. It is therefore

$$z_d = \frac{1}{\dfrac{1}{z_a} + \dfrac{1}{z_b}} = \frac{1}{\dfrac{1}{\dfrac{1}{i\omega m}} + \dfrac{1}{\dfrac{i\omega}{k}}} = \frac{1}{i\omega m + \dfrac{k}{i\omega}}$$

$$= \frac{i}{\dfrac{k}{\omega} - m\omega} \qquad [9\cdot 22]$$

The presence of i indicates that no energy is dissipated. Now if the force is known the velocity may be determined directly from the definition of the mobility of a point so that

$$v_d = Fz_d$$

Thus, with the velocity known, the displacement and the acceleration may be determined by converting according to the interrelations existing among the displacement, velocity, and acceleration as listed in Table 9·1. This simplifies the analysis of the motion of any point in the system. Resonance curves, such as are shown in Fig. 3·14, may be found by substituting a series of values for ω. The point of maximum amplitude gives the resonant speed. Another way of finding the resonant speed when there is no damping is to recognize the fact that the system is infinitely mobile at this speed. That is, theoretically it should take no force to maintain the motion. Thus, with infinite mobility the denominator of the expression for the mobility should be zero. This gives

$$\frac{k}{\omega} - m\omega = 0$$

or

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{W}}$$

which is the expression for the natural frequency derived in Chapter II.

When the support is moved as in Fig. 9·2 the analysis follows a similar method. The mobility of point b where the force is applied is equal to the sum of the mobilities of the spring and the mass because they are acting in series. Therefore the mobility of b is given as

$$z_b = z_a + z_{ba} = \frac{1}{i\omega m} + \frac{i\omega}{k} \qquad [9\cdot 23]$$

The velocity of point b can be found from the definition of the mobility. This is

$$v_b = z_b F_b = F_b \left(\frac{\omega}{k} - \frac{1}{\omega m}\right) i \qquad [9\cdot 24]$$

We are more interested in the motion of a because it is a point on the mass. Therefore, it is necessary to work back. The velocity at a is determined from the fundamental definition of mobility, which gives

$$v_a = F_a z_a = F_a \frac{1}{i\omega m} \qquad [9\cdot 25]$$

Since the force F_a must equal F_b, the resulting motion of a is easily obtained. To obtain a general expression for the resonant frequency it is necessary to express the velocity of a in terms of the velocity of b because it depends on it. This may be done by expressing F_a in terms of F_b as determined from equation 9·24. Substituting this in equation 9·25, we have

$$v_a = \frac{1}{i\omega m}\left(\frac{v_b}{i\left(\frac{\omega}{k} - \frac{1}{\omega m}\right)}\right) = \frac{-v_b}{\frac{\omega^2 m}{k} - 1}$$

At resonance the velocity v_a should theoretically approach infinity, which means that for any finite value of v_b the denominator must be zero. This expression when set equal to zero is

$$\frac{\omega^2 m}{k} - 1 = 0$$

or

$$\omega = \sqrt{\frac{k}{m}}$$

This, too, is the expression for frequency previously given in Chapter II. These two examples show that in order to solve a free vibration

problem it is necessary to supply a force and then determine the conditions that will produce infinite mobility or velocity.

A two-mass system with damping as shown in Fig. 9·7*a* will be used to illustrate the solution of a more complex system. When damping is included it is usually easier to use specific numerical values for k, m, r, F, and ω. General solutions may become involved. The values for the system are

$$\begin{aligned}
k_1 &= 20 \text{ lb/in.}\\
k_2 &= 50 \text{ lb/in.}\\
r_1 &= 2.5 \text{ lb-sec/in.}\\
r_2 &= 1.25 \text{ lb-sec/in.}\\
m_1 &= 19.3 \text{ lb} = 0.05 \text{ lb-sec}^2\text{/in.}\\
m_2 &= 30.9 \text{ lb} = 0.08 \text{ lb-sec}^2\text{/in.}\\
F_0 &= 7 \text{ lb at } 1760 \text{ rpm or } 184 \text{ radians/sec}\\
&\ \text{applied to mass 2}
\end{aligned}$$

The diagram is shown in Fig. 9·7*b*. The mobilities of the various points may now be calculated. Again the work must proceed toward the impressed force.

$$z_b = \frac{i\omega}{k} = \frac{i184}{20} = 9.2i$$

$$z_c = \frac{1}{r_1} = \frac{1}{2.5} = 0.4$$

$$z_d = \frac{-i}{\omega m_1} = \frac{-i}{184(0.05)} = -0.109i$$

Each of these three members acts relative to the ground so that the mobility of point e is

$$z_e = \cfrac{1}{\cfrac{1}{9.2i} + \cfrac{1}{0.4} + \cfrac{1}{-0.109i}} = 0.0281 - 0.1022i$$

$$z_{fe} = \frac{i\omega}{k} = \frac{184}{50}i = 3.68i$$

$$z_f = z_{fe} + z_e = 3.68i - 0.1022i + 0.0281 = 3.58i + 0.0281$$

$$z_g = \frac{1}{1.25} = 0.8$$

$$z_h = \frac{-i}{\omega m} = \frac{-i}{184(0.08)} = -0.068i$$

$$z_n = \cfrac{1}{\cfrac{1}{0.0281 + 3.58i} + \cfrac{1}{0.8} + \cfrac{1}{-0.068i}} = 0.006 - 0.069i$$

$$= 0.0692e^{i(-85°)}$$

With the mobility and force known, the velocity may be calculated.

$$v_n = z_n F_0 = (0.006 - 0.069i)7 = 0.042 - 0.483i$$

$$= 0.484e^{i(-85°)} = 0.484 \measuredangle\!\!\!-\, 85°$$

This indicates that the maximum velocity of mass 2 is 0.484 in./sec and that the velocity lags the force by 85°. The symbol ⦣ is used to indicate negative angles, and the symbol ∠ positive angles. This analysis may be carried back through the system and the motion of all other parts may be determined.

$$v_f = v_n$$

$$F_f = \frac{v_f}{z_f} = \frac{0.042 - 0.483i}{0.0281 + 3.58i} = -0.135 - 0.0128i$$

$$= 0.136 ⦣ 174.7°$$

The amplitude of the force at f then equals 0.136 lb and lags the impressed force by 174.7°. Now then

$$v_e = F_e z_e = F_f z_e = (-0.135 - 0.0128i)(0.0281 - 0.1022i)$$

$$= 0.0051 + 0.0134i = 0.0143 \angle 69.2°$$

and

$$v_b = v_e = v_d = v_c$$

The amplitudes may be obtained from the fundamental relationship

$$x = \frac{v}{i\omega} = \frac{v_h}{i\omega} \text{ or } \frac{v_d}{i\omega}$$

The displacements are 90° behind and are given as

$$x_2 = \frac{0.484 ⦣ 85°}{184i} = 0.00263 ⦣ 175° \text{ in.}$$

$$x_1 = \frac{0.0143 \angle 69.2°}{184i} = 0.000078 ⦣ 20.8° \text{ in.}$$

The mobility method works equally well for torsional vibration problems. The application to a torsional system is precisely the same as for a linear vibration. A three-mass system, as in Fig. 9·9a, would be represented by a schematic diagram as shown in Fig. 9·9b.

The mobility of the points may now be determined.

$$z_b = z_a + z_{ba}$$

$$z_d = \cfrac{1}{\cfrac{1}{z} + \cfrac{1}{z_a + z_{ba}}}$$

$$z_e = z_{ed} + \cfrac{1}{\cfrac{1}{z_c} + \cfrac{1}{z_a + z_{ba}}}$$

$$z_g = \cfrac{1}{\cfrac{1}{z_f} + \cfrac{1}{z_{ed} + \cfrac{1}{\cfrac{1}{z_c} + \cfrac{1}{z_{ba} + z_a}}}}$$

$$= \cfrac{1}{i\omega I_1 + \cfrac{1}{\cfrac{i\omega}{k_{t1}} + \cfrac{1}{i\omega I_2 + \cfrac{1}{\cfrac{i\omega}{k_{t2}} + \cfrac{1}{i\omega I_3}}}}}$$

Notice that this equation is a continuous repeating fraction. The solution to this is rather long but direct. Using values for a specific problem proves to be easier than the solution for a general expression. The solution may be simplified in this case by reducing the i so that the preceding equation becomes

$$z_g = \cfrac{-i}{\omega I_1 - \cfrac{1}{\cfrac{\omega}{k_{t1}} - \cfrac{1}{\omega I_2 - \cfrac{1}{\cfrac{\omega}{k_{t2}} - \cfrac{1}{\omega I_3}}}}}$$

This equation becomes

$$z = \frac{-i\left[\omega^4 \dfrac{I_1 I_2}{k_{t1}k_{t2}} - \omega^2\left(\dfrac{I_1 + I_2}{k_{t2}} + \dfrac{I_1}{k_{t1}}\right) + 1\right]}{\omega\left[\omega^4 \dfrac{I_1 I_2 I_3}{k_{t1}k_{t2}} - \omega^2\left(\dfrac{I_1 I_3 + I_1 I_2}{k_{t1}} + \dfrac{I_1 I_3 + I_2 I_3}{k_{t2}}\right) + I_1 + I_2 + I_3\right]} \quad [9 \cdot 26]$$

This is the general solution to the equation for three masses in torsional vibration without damping. The critical speeds may be obtained by setting the denominator equal to zero. This expression can be recognized as the equation usually given for a three-mass system. (See equation 4·23.)

When more than three masses are considered, the solution becomes more complicated. The solution of these systems can be obtained by using the results of an analysis based on the mobility method. For a free torsional system, the equation is

$$z = \cfrac{-i}{\omega I_n - \cfrac{1}{\cfrac{\omega}{k_{tn-1}} - \cfrac{1}{\omega I_{n-1} - \cfrac{1}{\cfrac{\omega}{k_{tn-2}} - \cfrac{1}{\omega I_{n-2} - \cfrac{1}{\cfrac{\omega}{k_{tn-3}} - \cdots}}}}}} \quad [9 \cdot 27]$$

where n is the number of masses. This may be reduced to a numerator and denominator. If the denominator is set equal to zero, the equation may be solved for the values of ω to get the critical speeds in radians per second.

9·7. Branched Torsional System. The solution of branched systems by trial and error methods is involved. For the simpler forms of branched systems it may be advantageous to set up a general solution for calculating the natural frequencies, particularly if the system is often used. The mobility method provides a means which involves only algebraic operations. The reduced system shown in Fig. 9·12*a*, which represents a simple branched propeller drive, will be analyzed to obtain a general solution for the critical speeds and then for amplitudes at one particular frequency.

After the mobility diagram, Fig. 9·12b, is drawn, an expression for the mobility at terminal n may be written as

$$z_n = \cfrac{1}{\cfrac{1}{z_h} + \cfrac{1}{z_{gf} + \cfrac{1}{\cfrac{1}{z_e} + \cfrac{1}{z_{ba} + z_a} + \cfrac{1}{z_{dc} + z_c}}}}$$

This can be reduced to a single fraction, which is

$$z_n = \frac{z_h(z_{gf} + z_e)(z_{ba} + z_a)(z_{dc} + z_c) + z_{gf}z_e(z_{dc} + z_c + z_{ba} + z_a)}{(z_{gf} + z_e + z_h)(z_{ba} + z_a)(z_{dc} + z_c) + z_e(z_{gf} + z_h)(z_{dc} + z_c + z_{ba} + z_a)}$$

The denominator of this equation may be set equal to zero in order to get the condition for the natural frequencies of the system. Thus, the equation for the natural frequencies is

$$(z_{gf} + z_e + z_h)(z_{ba} + z_a)(z_{dc} + z_c) + z_e(z_{gf} + z_h)(z_{dc} + z_c + z_b + z_a) = 0 \qquad [9\cdot 28]$$

The values of the individual mobilities must be substituted to get the equation in terms of the physical quantities of the system and the frequency. The values of the individual mobilities are

$$z_a = \frac{1}{i\omega I_1} \qquad z_{ba} = \frac{i\omega}{k_{t1}}$$

$$z_c = \frac{1}{i\omega I_2} \qquad z_{dc} = \frac{i\omega}{k_{t2}}$$

$$z_e = \frac{1}{i\omega I_3} \qquad z_{gf} = \frac{i\omega}{k_{t3}}$$

$$z_h = \frac{1}{i\omega I_4}$$

These values may be substituted in equation 9·28, and the whole reduced to

$$-I_1I_2I_3I_4\omega^6 + \left[\frac{I_2I_4}{k_{t2}k_{t3}}(I_1 + I_3) + \frac{I_1I_4}{k_{t1}k_{t3}}(I_2 + I_3) + \frac{I_1I_2}{k_{t1}k_{t2}}(I_3 + I_4)\right]\omega^4$$
$$- \left[\frac{I_1(I_2 + I_3 + I_4)}{k_{t1}} + \frac{I_2(I_1 + I_3 + I_4)}{k_{t2}} + \frac{I_4(I_1 + I_2 + I_3)}{k_{t3}}\right]\omega^2 +$$
$$I_1 + I_2 + I_3 + I_4 = 0 \qquad [9\cdot 29]$$

This is a cubic equation in ω^2 which can be readily solved when the numerical values of I and k are introduced.

If the behavior of the whole system is desired at any particular frequency, considerably less algebraic manipulation is involved provided the numerical values are introduced at the start. To illustrate the procedure let us assume a system wherein

$$I_1 = 2000 \text{ lb-in.-sec}^2$$

$$I_2 = 3000 \text{ lb-in.-sec}^2$$

$$I_3 = 1000 \text{ lb-in.-sec}^2$$

$$I_4 = 4000 \text{ lb-in.-sec}^2$$

$$k_{t1} = 600{,}000 \text{ lb-in./radian}$$

$$k_{t2} = 800{,}000 \text{ lb-in./radian}$$

$$k_{t3} = 500{,}000 \text{ lb-in./radian}$$

$$T_0 = 1000 \text{ lb-in.}$$

$$\omega = 23 \text{ radians/sec (220 rpm)}$$

From these values the mobilities of the members in the system may be obtained. They are

$$z_a = \frac{1}{i(23)2000} = -0.0000218i$$

$$z_c = \frac{1}{i(23)3000} = -0.0000145i$$

$$z_e = \frac{1}{i(23)1000} = -0.0000435i$$

$$z_h = \frac{1}{i(23)4000} = -0.00001087i$$

$$z_{ba} = \frac{i(23)}{600{,}000} = 0.0000383i$$

$$z_{dc} = \frac{i(23)}{800{,}000} = 0.0000288i$$

$$z_{gf} = \frac{i(23)}{500{,}000} = 0.0000460i$$

$$z_d = 0.0000288i - 0.0000145i = 0.0000143i$$

$$z_b = 0.0000383i - 0.0000218i = 0.0000165i$$

$$z_f = \cfrac{1}{\cfrac{10^5}{1.43i} + \cfrac{10^5}{1.65i} - \cfrac{10^5}{4.35i}} = 0.0000093i$$

$$z_g = 0.0000093i + 0.0000460i = 0.0000553i$$

$$z_n = \cfrac{1}{\cfrac{10^5}{5.53i} - \cfrac{10^5}{1.087i}} = -0.0000135i$$

From the fundamental relation $z = \Omega/T$, the angular velocity Ω_n is

$$\Omega_n = z_n T_n = -0.0000135i(1000) = -0.0135i$$

$$\Omega_n = \Omega_h = \Omega_g$$

Since the mobility and angular velocity of g are known, the torque on g is

$$T_g = \frac{\Omega_g}{z_g} = \frac{-0.0135i}{0.0000553i} = -244$$

$$T_f = T_g$$

$$\Omega_f = z_f T_f = 0.0000093i(-244) = -0.00227i$$

$$\Omega_b = \Omega_d = \Omega_e = \Omega_f$$

$$T_b = \frac{\Omega_b}{z_b} = \frac{-0.00227i}{0.0000165i} = -137.5 = T_a$$

$$T_d = \frac{\Omega_d}{z_d} = \frac{-0.00227i}{0.0000143i} = -159 = T_c$$

$$T_e = \frac{\Omega_e}{z_e} = \frac{-0.00227i}{-0.0000435i} = 52.5$$

It is of interest to note that

$$T_b + T_d + T_e = -244$$

as it should for this torque has been divided between the masses. The torque on I_4 at h is

$$T_h = \frac{\Omega_h}{z_h} = \frac{-0.0135i}{-0.00001007i} = 1244$$

This checks also because

$$T_h + T_g = 1000$$

The angular velocities at the masses may be determined from the mobility.

$$\Omega_a = z_a T_a = -0.0000218i(-137.5) = 0.003i$$

$$\Omega_c = z_c T_c = -0.0000145i(-159) = 0.0023i$$

$$\Omega_e = z_e T_e = -0.0000435i(52.2) = -0.00227i$$

$$\Omega_h = -0.0135i$$

The angular displacements may be determined from the angular velocity and the relation

$$\theta = \frac{\Omega}{i\omega} = \frac{\Omega}{i23}$$

The results are

$$\theta_a = 0.00013 \text{ radian}$$

$$\theta_c = 0.00010 \text{ radian}$$

$$\theta_e = -0.000099 \text{ radian}$$

$$\theta_h = -0.0000587 \text{ radian}$$

These values are all for a disturbing torque of 1000 lb-in. The angular displacement or velocity for any other torque is obtained by direct proportion.

9·8. Determination of Stresses. The preceding sections gave a method of determining the forces or torques acting through any member in a vibrating system. When these values are known it is a simple matter to solve for the stresses in these same members. In the example of the previous section the torque on the shaft with a torsional spring constant of 800,000 lb-in./radian was 149 lb in. The shear stress due to the vibration would be given by the equation

$$s_s = \frac{16T_v}{\pi d^3}$$

where s_s = torsional shear stress due to vibration in pounds per square inch
T_v = torque on shaft due to vibration in pound-inches
d = diameter of the actual shaft in inches

Since this is only the vibrational stress it must be added to the torsional stress resulting from the load.

In order to make a complete analysis of a system, it is necessary to take into account the nature of the disturbing force or moment. This requires an harmonic analysis of the disturbing force or moment to find which component frequencies have sufficient magnitude to cause serious vibration. Frequencies near the natural frequency or which are simple fractions or multiples of the natural frequency are generally most dangerous.

PROBLEMS

9·1. (*a*) Draw the mobility diagram for the system shown in Fig. P9·1.

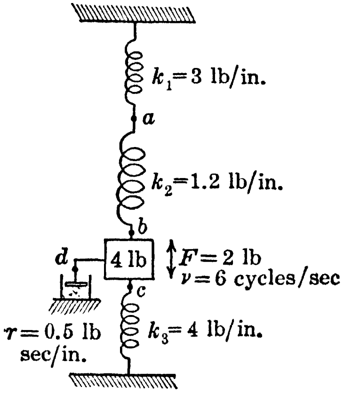


FIG. P9·1.

(*b*) Determine the mobilities for the system, and from them find the velocity and amplitude of the mass.

9·2. (*a*) Draw the mobility diagram for the two-disk and two-shaft system shown in Fig. P9·2.

(*b*) Determine the equation for the natural frequencies for this system.

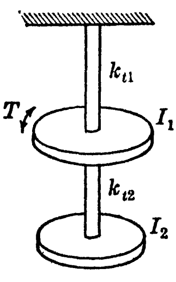


FIG. P9·2.

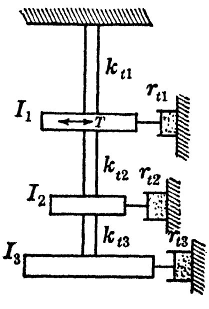


FIG. P9·3.

9·3. (*a*) Draw the mobility diagram for the three-disk system shown in Fig. P9·3.

(*b*) Draw the mobility diagram for the same system except that there is damping only between the disks.

9·4. Find the equation for the natural frequencies of a four-disk and three-shaft system with no damping.

9·5. Find the equation for the natural frequency of a two-disk and one-shaft system using the mobility method.

9·6. In problem 4·3 the first disk I_1 is acted upon by an harmonic torque of 2000 lb-in. at a rate of 180 cycles/min. What is the amplitude of displacement for each disk? What is the stress due to vibration in each shaft?

9·7. An harmonic torque of 2000 lb-in. acts on the center disk of problem 4·3 at a rate fo 180 cycles/min.

(*a*) Set up the mobility diagram for this system.

(*b*) Determine the amplitudes of displacement for the three disks.

(*c*) Determine the vibrational stresses in the two shafts.

9·8. In problem 4·6 assume that an harmonic torque of 1800 lb-in. at a frequency of 2600 cycles/min acts on the crank.

(*a*) What are the amplitudes of motion due to the vibration?

(*b*) What is the vibration stress in the steel shaft between the crank and the propeller? It has a 3-in. outside diameter and a $2\frac{1}{4}$-in. inside diameter.

9·9. (*a*) Draw the mobility diagram for the system shown in Fig. P9·9.

(*b*) Determine the mobility of mass 1 when $F = 2$ lb and $\mu = 10$ cycles/sec.

(*c*) Determine the amplitude of mass 1.

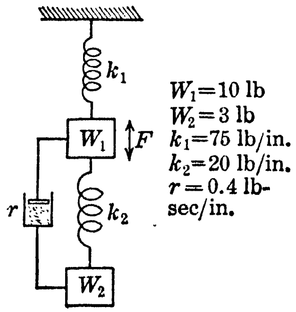


FIG. P9·9.

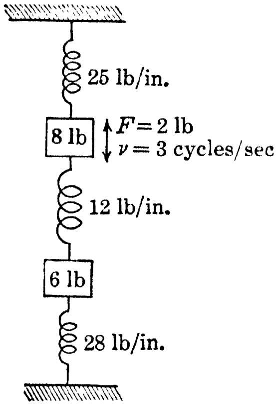


FIG. P9·10.

9·10. (*a*) Draw the mobility diagram for the system shown in Fig. P9·10.

(*b*) Find the equation from which the natural frequencies may be determined.

(*c*) Find the amplitudes of the masses.

CHAPTER X

MECHANICAL AND ELECTRICAL MODELS OF VIBRATION SYSTEMS

10·1. Introduction. The direct mathematical solution of many engineering problems is frequently so complicated that a direct or mathematical answer cannot be obtained. If experimental methods are to be applied, the scope of the results can frequently be increased through the use of dimensional analysis. There is, in addition to the dimensional analysis approach, the very important fields of analogies and models which may be drawn upon for the solution of many problems.[53, 54, 55] The fact that many different physical phenomena can be represented by similar or equivalent equations may permit the results found in one field of science to be applied or used in some other field. In some fields of science, experimental work can be carried out conveniently and economically; in others, experimental work may be difficult or impossible. This general discussion will consider, first, dimensional analysis and its application to mechanical models and, second, the possible uses of electrical models for the solution of some types of mechanical vibration problems.

10·2. Dimensional Analysis. One method frequently used in the analysis of data and in the design of models of mechanical systems is dimensional analysis. Dimensional analysis employs the principle that any equation representing physical phenomena must be dimensionally homogeneous. This method of analysis is useful if some knowledge of the problem being solved is available. It is not satisfactory to start out with a long list of possible quantities which might enter into the problem and expect results of any value. Bridgeman says in his book:[56] "The man applying dimensional analysis is not to ask himself 'On what quantities does the result depend?' for this question gets nowhere and is not pertinent. Instead we are to imagine ourselves as writing out the equations of motion with sufficient detail to be able to enumerate the elements which enter them."

The procedure in the application of the method of dimensional analysis is, first, to list the various fundamental physical factors or dimensions which enter into the problem as determined from the equations of

motion of the system under consideration. The fundamental dimensions of each of these are then set down. The π theorem, as developed by Buckingham, is that, if the number of independent factors is n and the number of fundamental dimensions involved is p, there will be $(n - p)$ dimensionless parameters, or π functions, involved in the solution of the problem.[57] In the general analysis of vibration problems the units generally employed are force in pounds F, length in inches L, and time in seconds T.

For the problem at hand the quantities that may reasonably affect the results can be determined by examining the differential equations of the simpler problems. These and their fundamental units are

$$\text{Mass—}m = \frac{W}{g} - \frac{FT^2}{L}$$

$$\text{Spring constant—}k - \frac{F}{L}$$

$$\text{Damping constant—}r - \frac{FT}{L}$$

$$\text{Natural frequency—}\omega - \frac{1}{T}$$

$$\text{Forced frequency—}\nu - \frac{1}{T}$$

$$\text{Force—}F\text{—}F$$

$$\text{Displacement—}x\text{—}L$$

It will be noted that the term mass has been introduced in place of W/g. This simplifies the handling of dimensional analysis and will be used in this chapter. The time might be considered, but it does not seem necessary because the motion is periodic. Thus the time is really the reciprocal of the frequency and need not be included. The length of spring or any other physical dimension of the system does not enter in because the spring constant, mass, and damping constant would include any effects a length would have on the system.

The terms involve three fundamental units, F, T, and L. The number of π functions that may be determined is the number of factors (7) minus the number of fundamental units (3), or (4) π functions. The procedure in the application of the π theorem of Buckingham is to select from among the variables listed three which contain all three of the fundamental dimensions among them. These are set down, with one of each of the other variables in turn written after them. Each of them

will, upon further analysis, yield a dimensionless quantity. The form of this quantity is determined by solving for the powers of each quantity affecting the problem. This is done by setting the sum of the exponents of each fundamental unit equal to zero and then solving simultaneously for each exponent. If a, b, and c are arbitrary powers to be assigned to the three quantities already selected, then the dimensional equations can be written as

$$\pi_1 = m^a x^b \omega^c k = \left(\frac{FT^2}{L}\right)^a (L)^b \left(\frac{1}{T}\right)^c \frac{F}{L}$$

$$\pi_2 = m^a x^b \omega^c r = \left(\frac{FT^2}{L}\right)^a (L)^b \left(\frac{1}{T}\right)^c \frac{FT}{L}$$

$$\pi_3 = m^a x^b \omega^c \nu = \left(\frac{FT^2}{L}\right)^a (L)^b \left(\frac{1}{T}\right)^c \frac{1}{T}$$

$$\pi_4 = m^a x^b \omega^c F = \left(\frac{FT^2}{L}\right)^a (L)^b \left(\frac{1}{T}\right)^c F$$

The sum of the exponents for each of the fundamental units must be zero. Therefore for π_1

$$a + 1 = 0 \quad \text{(Powers of force, } F\text{)}$$

$$2a - c = 0 \quad \text{(Powers of time, } T\text{)}$$

$$-a + b - 1 = 0 \quad \text{(Powers of length, } L\text{)}$$

When these equations are solved simultaneously the values of the exponents are

$$a = -1$$

$$b = 0$$

$$c = -2$$

so that the first dimensionless function, π_1, is

$$\pi_1 = \frac{k}{m\omega^2} = \frac{kg}{W\omega^2} \qquad [10\cdot1]$$

The remaining functions may be evaluated in a like manner to obtain

$$\pi_2 = \frac{r}{m\omega} = \frac{rg}{W\omega} \qquad [10\cdot2]$$

$$\pi_3 = \frac{\nu}{\omega} \qquad [10\cdot3]$$

$$\pi_4 = \frac{F}{mx\omega^2} = \frac{Fg}{Wx\omega^2} \qquad [10\cdot4]$$

If fewer quantities were assumed at the beginning, either fewer dimensionless functions would be found or else it would be impossible to evaluate them. This is particularly true if x is omitted.

10·3. Mechanical Models. Models of vibrating systems may be made just as easily as in some other fields of engineering, such as fluid mechanics and stress analysis. It is necessary to retain certain dimensionless quantities just as in any other model making. Two alternatives in determining these dimensionless terms are available. The first is to apply the methods of dimensional analysis. The more definite but less well-known method is changing variables in the differential equations.

When the differential equations of motion of a system can be written, dimensionless constants that fix the characteristics of the system may be determined directly from the equation by putting the equation in dimensionless form. This can be accomplished mathematically by changing to dimensionless variables. Equation 10·5, representing a simple forced vibration system of one degree of freedom, will be used to illustrate the method of changing variables.

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = F\sin\nu t \qquad [10\cdot5]$$

If a change of variable is made using

$$X = \frac{x}{L}$$

and

$$T = \omega t$$

where L is some characteristic length and ω is some angular velocity in radians per second. When the two dimensionless terms X and T are differentiated, the results are

$$dx = LdX$$

$$dt = \frac{dT}{\omega}$$

so that

$$\frac{dx}{dt} = L\omega\frac{dX}{dT}$$

This first derivative may be differentiated again with respect to t so that

$$\frac{d^2x}{dt^2} = L\omega\frac{d^2X}{dtdT}\left(\frac{\omega dt}{dT}\right) = L\omega^2\frac{d^2X}{dT^2}$$

When these derivatives are substituted in the original equation the result is

$$mL\omega^2 \frac{d^2X}{dT^2} + rL\omega \frac{dX}{dT} + kLX = F \sin \frac{\nu T}{\omega} \qquad [10 \cdot 6]$$

Or, if it is rearranged,

$$\frac{d^2X}{dT^2} + \left[\frac{r}{m\omega}\right] \frac{dX}{dT} + \left[\frac{k}{m\omega^2}\right] X = \left[\frac{F}{mL\omega^2}\right] \sin \left[\frac{\nu}{\omega}\right] T \qquad [10 \cdot 7]$$

Now, since X and T are dimensionless, their derivatives are dimensionless. This makes each term in equation 10·7 dimensionless. From this it follows that $r/m\omega$, $k/m\omega^2$, $F/mL\omega^2$, and ν/ω are all dimensionless characteristics for a forced vibration problem, and all are identical with equations 10·1, 10·2, 10·3, and 10·4 when $L = x$. The first two terms determine the physical characteristics of the system whether it is forced or free vibration. The last two terms are used only for forced vibration because F and ν disappear in free vibration.

When a model for a mechanical system with more than one mass is to be made, it is necessary to change variables in all the equations. Then each equation will yield a set of dimensionless constants that must be maintained. These constants will be of the form of any or all the previous constants.

To build a model of a mechanical vibrating system it is necessary only for the dimensionless constants for the model to equal the respective constants for the prototype. If it is desired to determine the natural frequency, only the first two constants listed need be considered. When more qualitative measurements are desired, however, it will be necessary to consider all the terms. It should be noted that these dimensionless terms can take many different forms, depending upon which coefficient in equation 10·6 is used as the divisor. The form which should be used will depend to a great extent upon the problem. If F were used as a divisor so that all the dimensionless terms included F, the form would be of no value for free vibrations. The form given by equation 10·7 is usually the most useful.

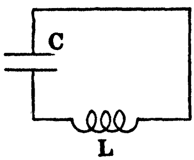


FIG. 10·1.

10·4. Equations for Electric Circuits. Differential equations may be written for electric circuits just as for mechanical systems. Several simple cases which will be used later are set up in the succeeding pages. If a capacitor with a capacitance **C** and a coil with an inductance **L** are connected in a series circuit, as shown in Fig. 10·1, Kirchhoff's laws may be used to write the equation for the electromotive force. The emf

for the inductance is given by $\mathbf{L}\dfrac{d\mathbf{I}}{dt}$ or $\mathbf{L}\dfrac{d^2\mathbf{Q}}{dt^2}$ where $\mathbf{I}$ is the current and $\mathbf{Q}$ is the charge. The emf for the capacitor is $\dfrac{1}{\mathbf{C}}\displaystyle\int \mathbf{I}dt$ or $\dfrac{1}{\mathbf{C}}\mathbf{Q}$. From Kirchhoff's law, then, the sum of the potentials must be zero so that we may write

$$\mathbf{L}\frac{d^2\mathbf{Q}}{dt^2} + \frac{1}{\mathbf{C}}\mathbf{Q} = 0 \qquad [10\cdot 8]$$

This equation of motion for the electric circuit is seen to be of the same form as that for a freely vibrating mechanical system with no damping.

If now some resistance is included in series with the capacitor and the inductance, as indicated in Fig. 10·2, a potential drop $\mathbf{IR}$ must be added in the equation so that the equation for the circuit becomes

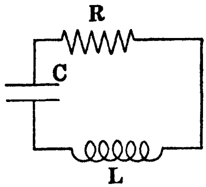


FIG. 10·2.

$$\mathbf{L}\frac{d^2\mathbf{Q}}{dt^2} + \mathbf{R}\frac{d\mathbf{Q}}{dt} + \frac{1}{\mathbf{C}}\mathbf{Q} = 0 \qquad [10\cdot 9]$$

because $\mathbf{I} = d\mathbf{Q}/dt$. This equation may be recognized as being similar in form to that for free vibration of mechanical system with damping.

When an alternating current is impressed upon the circuit, as represented by Fig. 10·3, the impressed voltage $\mathbf{E}\sin\phi t$ must be added to the circuit. The equation then for a forced electric circuit is

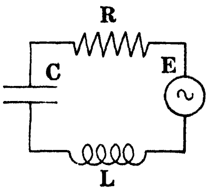


FIG. 10·3.

$$\mathbf{L}\frac{d^2\mathbf{Q}}{dt^2} + \mathbf{R}\frac{d\mathbf{Q}}{dt} + \frac{1}{\mathbf{C}}\mathbf{Q} = \mathbf{E}\sin\phi t \qquad [10\cdot 10]$$

This too may be recognized as the counterpart of the forced mechanical vibrating system.

10·5. Models of an Electric Circuit. Models may be made for an electric circuit just as for a mechanical system. The differential equation may be used again to give the dimensionless constants by changing variables. If $\mathbf{M}$ is a charge and Ω is a frequency, the variables

$$q = \frac{\mathbf{Q}}{\mathbf{M}}$$

$$T = \Omega t$$

and their derivatives may be substituted in equation 10·10 to give

$$\frac{d^2q}{dT^2} + \left[\frac{\mathbf{R}}{\mathbf{L}\Omega}\right]\frac{dq}{dT} + \left[\frac{1}{\mathbf{CL}\Omega^2}\right]q = \left[\frac{\mathbf{E}}{\mathbf{LM}\Omega^2}\right]\sin\left[\frac{\phi}{\Omega}\right]T \qquad [10\cdot 11]$$

Since q and T and their derivatives are dimensionless, the remaining terms are also dimensionless. Thus for a model it is necessary that $\mathbf{R}/\mathbf{L}\Omega$, $1/\mathbf{C}\mathbf{L}\Omega^2$, $\mathbf{E}/\mathbf{LM}\Omega^2$, and ϕ/Ω for the actual circuit be equal to the same terms for the original electric circuit.

A two-mesh circuit may be treated the same way; that is, each equation may be made dimensionless by changing variables. Then more dimensionless numbers are obtained and must be maintained for models.

10·6. Analogy between Mechanical and Electric Systems. Throughout the preceding sections it was noted that the electric and mechanical systems can have the same form in their differential equations. If equations 10·5 and 10·10 are written together, the analogy is more noticeable. Thus

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = F\sin\nu t \tag{10·5}$$

$$\mathbf{L}\frac{d^2\mathbf{Q}}{dt^2} + \mathbf{R}\frac{d\mathbf{Q}}{dt} + \frac{1}{\mathbf{C}}\mathbf{Q} = \mathbf{E}\sin\phi t \tag{10·10}$$

Mathematically both equations have the same solution; the dimensions, however, are entirely different. From the equations above we see that

x, the displacement, corresponds to $\mathbf{Q}$, the charge

$\frac{dx}{dt}$, the velocity, corresponds to $\frac{d\mathbf{Q}}{dt}$, the current

$\frac{d^2x}{dt^2}$, the acceleration, corresponds to $\frac{d^2\mathbf{Q}}{dt^2}$

$\frac{W}{g} = m$, the mass, corresponds to $\mathbf{L}$, the inductance

r, the damping factor, corresponds to $\mathbf{R}$, the resistance

k, the spring constant, corresponds to $\frac{1}{\mathbf{C}}$, the reciprocal of the capacitance

F, the force, corresponds to $\mathbf{E}$, the voltage

ν, the mechanical forced frequency, corresponds to ϕ, the electric forced frequency

This correspondence is maintained in the equations of any mechanical system and the equations of the analogous electric circuit.

It is necessary to maintain a consistent set of units in both the mechanical and the electrical equations. In the mechanical system the units that are most commonly used are inches, pounds, and seconds; in the electric system they are volts, amperes, and seconds. The damping factor is then pound-seconds per inch, whereas the resistance is volts per ampere. The mass is given in terms of pound (seconds)^2 per inch, whereas the inductance is given in henrys because a henry may be defined in terms of volts, amperes, and seconds. The spring constant is given as pounds per inch of deflection. The value of the capacitance is given in farads, a term which depends upon the voltage and current. All quantities and their units are summarized in Table 10·1.

TABLE 10·1

Mechanical System		Electric System	
m	Mass—lb-sec^2/in.	L	Inductance—henrys
I	Moment of inertia—lb-in.-sec^2		
k	Spring constant—lb/in.	$\frac{1}{C}$	Reciprocal of capacitance—farads
k_t	Spring constant—lb-in./radian		
r	Damping constant—lb-sec/in.	R	Resistance—ohms
r_t	Damping constant—lb-in.-sec/radian		
F	Force—lb	E	Voltage—volts
T	Torque—lb-in.		
x	Displacement—in.	Q	Charge—coulombs
θ	Angle—radian		
$\frac{dx}{dt}$	Velocity—in./sec	$\frac{dQ}{dt}$	Current—amperes
$\frac{d\theta}{dt}$	Angular velocity—radians/sec		
ν	Forced frequency—radians/sec	ϕ	Forced frequency—radians/sec
ν	Forced frequency—radians/sec		
$\frac{d^2x}{dt^2}$	Acceleration—in./sec^2	$\frac{d^2Q}{dt^2}$	
$\frac{d^2\theta}{dt^2}$	Angular acceleration—radians/sec^2		

10·7. Electrical Model of Mechanical Vibrating System. In previous sections we have seen how the equations of both systems are mathematically the same. This also holds for the dimensionless quantities

$$\frac{r}{m\omega} = \frac{\mathbf{R}}{\mathbf{L}\Omega}$$

$$\frac{k}{m\omega^2} = \frac{1}{\mathbf{CL}\Omega^2}$$

$$\frac{F}{mL\omega^2} = \frac{\mathbf{E}}{\mathbf{LM}\Omega^2}$$

$$\frac{\nu}{\omega} = \frac{\phi}{\Omega}$$

Since both equations are the same, the solutions are the same. For free vibrations without damping or resistance, the solutions are

$$x = x_0 \sin \sqrt{\frac{k}{m}}\, t$$

$$\mathbf{Q} = \mathbf{Q}_0 \sin \sqrt{\frac{1}{\mathbf{CL}}}\, t$$

where the natural frequencies are

$$\omega_n = \sqrt{\frac{k}{m}} \text{ radians/sec}$$

$$\Omega_n = \sqrt{\frac{1}{\mathbf{CL}}} \text{ radians/sec}$$

For free vibration with damping or resistance the solution can be

$$x = e^{-\frac{rt}{2m}} A \sin \sqrt{\frac{k}{m} - \left(\frac{r}{2m}\right)^2}\, t$$

$$\mathbf{Q} = e^{-\frac{\mathbf{R}t}{2\mathbf{L}}} B \sin \sqrt{\frac{1}{\mathbf{CL}} - \left(\frac{\mathbf{R}}{2\mathbf{L}}\right)^2}\, t$$

where A and B are constants. Here the damped resonant frequency is

$$\omega_{nd} = \sqrt{\frac{k}{m} - \left(\frac{r}{2m}\right)^2} \text{ radians/sec}$$

$$\Omega_{nd} = \sqrt{\frac{1}{\mathbf{CL}} - \left(\frac{\mathbf{R}}{2\mathbf{L}}\right)^2} \text{ radians/sec}$$

For forced vibrations in the steady state the solutions are

$$x = \frac{F \sin \nu t}{\sqrt{(k - m\nu^2)^2 + r^2\nu^2}}$$

$$\mathbf{Q} = \frac{\mathbf{E} \sin \phi t}{\sqrt{\left(\frac{1}{\mathbf{C}} - \mathbf{L}\phi^2\right)^2 + \mathbf{R}^2\phi^2}}$$

The frequencies of motion are the same as that of the forced frequency.

This close analogy indicates that an electrical circuit will react the same as a mechanical system. That is, it will reach a point of resonance at some frequency and the charge Q will reach a magnitude dependent upon the impressed force. Furthermore, it shows that by making the dimensionless constants of both systems equal it is possible to choose electrical units, and by suitable measurements determine the resonant mechanical frequency or the points of maximum amplitude. Likewise with further measurements it is possible to determine quantitatively actual movements in the mechanical system.

Since the resonant or natural frequency is dependent only upon the free vibrating system or circuit, the force or voltage and the impressed frequency cannot affect it. This means only that the first two dimensionless quantities must be equal. That is,

$$\frac{k}{m\omega^2} = \frac{1}{\mathbf{CL}\Omega^2}$$

$$\frac{r}{m\omega} = \frac{\mathbf{R}}{\mathbf{L}\Omega}$$

To determine such a model it is best to start out assuming that ω is some fraction or multiple of Ω. Then, if all the mechanical terms are known there are still three variables in the electrical constants. One term may be chosen arbitrarily, but this fixes the remaining two. By trial and error suitable values may be arrived at. When more quantitative measurements are desired the remaining constants must also be equal. These are

$$\frac{F}{mL\omega^2} = \frac{\mathbf{E}}{\mathbf{LM}\Omega^2}.$$

$$\frac{\nu}{\omega} = \frac{\phi}{\Omega}$$

Since ω and Ω occur in all constants, it would seem reasonable to assume that they represent the natual frequency, although in forced vibration they might also represent the impressed frequency. Actually it makes little difference except that the frequencies of the two systems must maintain a definite ratio between them. After this ratio is assumed and the system is established, as in the case of free vibration, the voltage may be determined.

10·8. Measuring Analogous Quantities. The charge across a capacitor, the current through a resistance, and the rate of change in current through the inductance coil are all proportional to the drop in voltage across their respective terminals. Since the charge corresponds to the displacement, the voltage across the capacitor indicates the displacement in the equivalent system. The voltage across the resistance indicates the velocity in the equivalent system, and the voltage across the inductance coil indicates the acceleration. Any one measurement may be made, depending upon what quantity is desired. All others may be obtained from this measurement by dividing or multiplying by the proper function of the frequency. Measurement on the capacitor or coil is usually limited to one value of the capacitor or coil. Whether or not they may be used satisfactorily for measurement depends upon the sensitivity of the instruments used. The resistance can be varied considerably without affecting the frequency measurements.

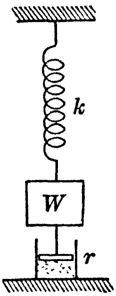


Fig. 10·4.

10·9. Method of Laying Out Equivalent Electric Circuits. In laying out equivalent electric circuits several generalities may be used to simplify the work. In a mechanical system, such as is shown in Fig. 10·4, the forces on the mass may be said to act in parallel relative to the frame. That is, the spring force depends upon the motion of the mass relative to the frame, the damping force depends upon the velocity of the mass relative to the frame, and the inertia force depends upon the acceleration of the mass relative to the frame. The forces, therefore, must be added when they act in parallel. The reverse also is true; that is, a force acts in series when it is transmitted through an element to another element. When the forces are in parallel, the electric elements in the equivalent system are put in series. Forces in series are represented by electric elements in parallel. These principles may be followed in the examples used previously or in the equivalent circuits shown in Figs. 10·5 to 10·10.

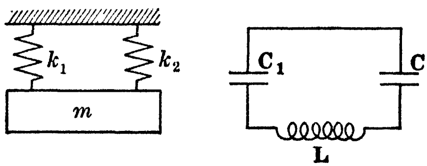


Fig. 10·5.

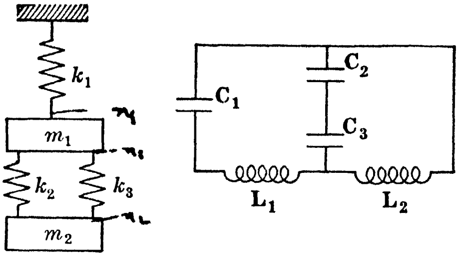


FIG. 10·6.

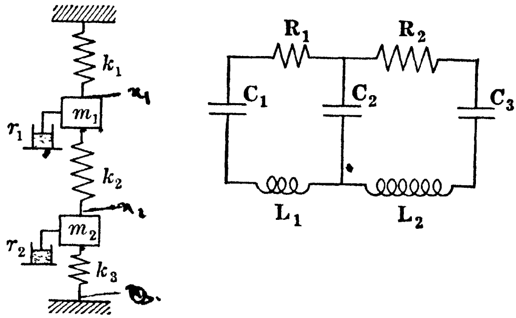


FIG. 10·7.

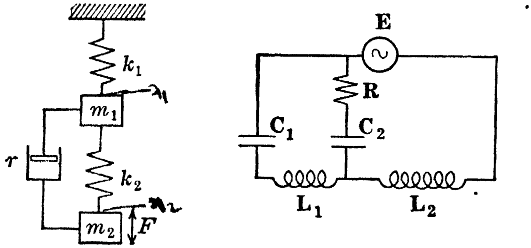


FIG. 10·8.

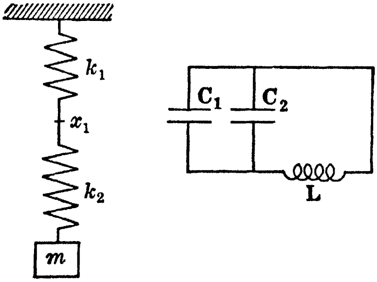


FIG. 10·9.

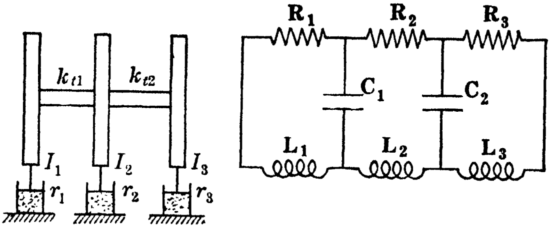


FIG. 10·10.

10·10. Application to Single-Mass System. The single-mass system with damping can be represented schematically, as shown by Fig. 10·4. The electrical model can be applied to this type of problem to obtain the natural frequency and also the relative amplitude of motion. The use of the method can best be shown by an example. Assume that the mechanical system shown in Fig. 10·4 has a weight of 10 lb suspended from a spring having a k value of 563 lb/in. The damping will be varied to study the effect of damping on the amplitude of motion. The critical damping of the system is

$$r_c = 2\sqrt{\frac{kw}{g}} = 2\sqrt{\frac{563(10)}{386}} = 7.65 \text{ lb-sec/in.}$$

The behavior of the system with r/r_c values of 0.03, 0.05, 0.10, 0.20, 0.30, and 0.50 is desired. For a value of 0.03 the mechanical resistance is $r = 0.03(7.65) = 0.229$ lb-sec/in.

In order to have equivalent systems it is necessary for the corresponding dimensionless quantities to be equal. The first sets to be satisfied are

$$\frac{\nu}{\omega} = \frac{\phi}{\Omega}$$

and

$$\frac{k}{m\omega^2} = \frac{1}{\mathbf{CL}\Omega^2}$$

If we assume that the electric frequencies are to be one hundred times the mechanical frequencies, the first relation is satisfied. The second equation reduces to

$$\frac{563(386)}{10\omega^2} = \frac{1}{\mathbf{CL}(100\omega)^2}$$

In order to obtain values of **C** and **L**, a value for one of them must be

assumed. If a value of **L** = 23 millihenrys is used, the value of **C** can be found to be

$$\mathbf{C} = \frac{10}{563(386)0.023(100)^2} = 0.2(10^{-6}) = 0.2 \text{ microfarad}$$

The amount of damping required must be determined from the other relation

$$\frac{rg}{W\omega} = \frac{\mathbf{R}}{\mathbf{L}\Omega}$$

$$\mathbf{R} = \frac{\mathbf{L}rg\ \Omega}{W\ \omega}$$

for

$$\frac{r}{r_c} = 0.03$$

or

$$r = 0.229 \text{ lb-sec/in.}$$

$$\mathbf{R} = \frac{0.023(0.229)386(100)}{10} = 20.4 \text{ ohms}$$

The electrical resistance for any other value of damping is in direct proportion to the damping constant. For a value of r/r_c = 0.10, **R** = 68 ohms. When an impressed alternating voltage is applied to the electric circuit it oscillates in the same manner as the equivalent mechanical system. Its behavior is most easily studied by finding the magnitude of the current flowing. This can be readily determined by measuring the voltage drop across a fixed resistance in the circuit. The current is then proportional to the velocity in the mechanical system. The relative displacement can be found by dividing the velocity by the frequency. This corresponds to using the methods developed in the previous chapter on mobility. The results with different amounts of damping are shown in Fig. 10·12, which gives a plot of relative velocity against frequency. Figure 10·13 shows the corresponding relative displacements with various damping. We see that the results correspond to those previously obtained from a theoretical solution. (See Fig. 3·10.)

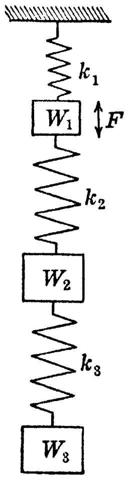


FIG. 10·11.

The natural frequency can also be obtained from such a model. The frequency values plotted in Figs. 10·12 and 10·13 are one one-hundredth of the values obtained from the electric circuit. The natural frequency

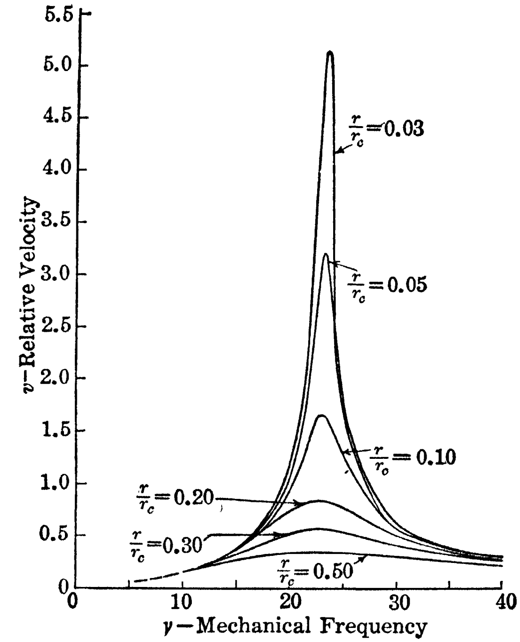


FIG. 10·12.

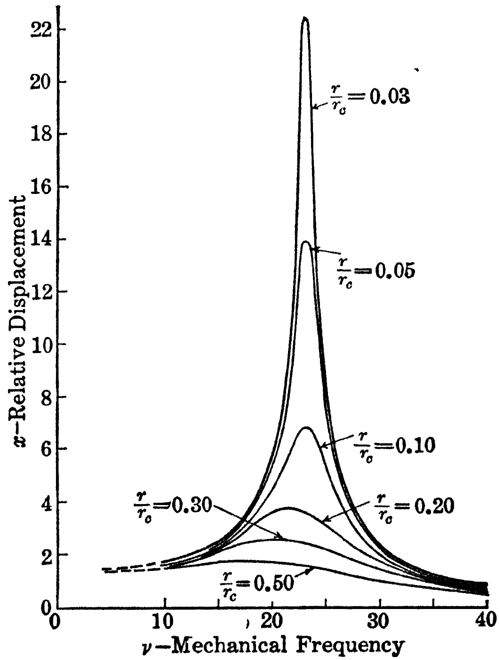


FIG. 10·13.

shown in these figures is 23 cycles/sec, whereas the value calculated is 23.4 cycles/sec.

10·11. Application to Multi-Mass Systems. Figure 10·14 shows the results of an analysis made on a three-mass system using the equivalent

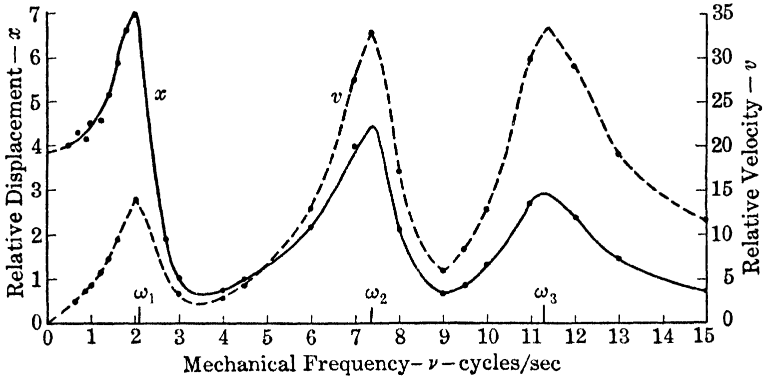


FIG. 10·14.

system shown in Fig. 10·11. The values used where $W_1 = 29$ lb, $W_2 = 57$ lb, $W_3 = 56$ lb, $k_1 = 90.5$, $k_2 = 181$, and $k_3 = 181$ lb/in. The corresponding electrical quantities were found as in the previous section. The ratio between mechanical and electric frequencies was taken as

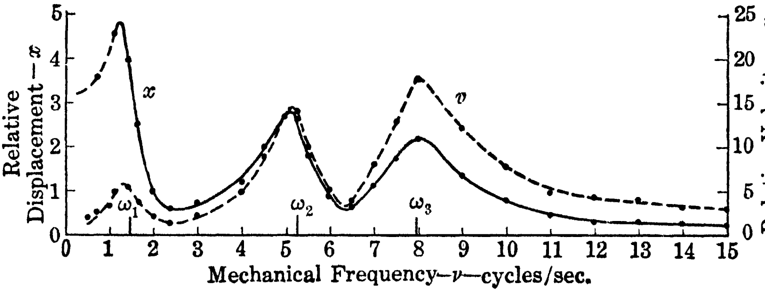


FIG. 10·15.

1 to 1000. The corresponding electrical values are $\mathbf{L}_1 = 41.5$, $\mathbf{L}_2 = 81.7$, $\mathbf{L}_3 = 80.2$ millihenrys, $\mathbf{C}_1 = 0.2$, $\mathbf{C}_2 = 0.01$, $\mathbf{C}_3 = 0.01$ microfarad. In this case only the natural frequenices were of interest so that the relative damping constants did not have to be determined. The calculated frequencies of the mechanical system are indicated as ω_1, ω_2, and ω_3. The agreement between calculated and measured values is good.

One advantage of the electrical model is that it makes it easy to determine what might be done to change a particular natural frequency so that it does not lie in the range of operating speeds. All that is necessary to determine the effect of any other mass or elasticity in the mechanical system is to change the corresponding electrical quantity and determine the new resonant frequency. If a variable frequency oscillator is used to give the impressed voltage, it takes only a few seconds to find out how a given change will affect the frequency or the relative amplitude. For example, Fig. 10·15 shows how doubling all the masses affects the natural frequency and relative amplitude of the system. The calculated frequencies are again indicated on the lower scale. The extension of this method offers many possibilities in the solution of multi-mass systems.

10·12. Experimental Equipment. The principles of the electrical analog may be demonstrated with very readily available equipment. General laboratory instruments may be used to make a good analysis. The inductances can be made of magnet wire wound on wooden spools. These values are fixed and can be wound and calibrated on a bridge such as the General Radio Company type 650A. An alternative is to check the resonant frequency with known capacitors. Winding must be done if very low resistance inductances are needed. Fixed inductors can also be purchased but their resistance tends to be high. If more work is contemplated it is desirable to purchase variable inductors. These can be purchased for a nominal amount in a decade of continuously variable type and have the advantage of having a good calibration.

Fixed capacitors can be bought at any radio supply store at a very low price. It is necessary to calibrate each capacitor as their accuracy is poor. A bridge may be used, or the resonant frequency can be checked with known inductances. Several capacitors may be combined to give intermediate values. Continuously variable and decade capacitors are also available at a reasonable price. These are calibrated with fairly good accuracy. Fixed resistors can also be bought in a radio supply store. Decades or rheostats can be used to secure a variable resistance. The resistance of the inductors must be allowed for in designing any damping.

Several good audio oscillators are now available. It is desirable to have a unit which retains its calibration for long times and one that has a frequency range from about 20 to 100,000 cycles/sec. A smaller range can be used if care is exercised in keeping within this frequency range.

It is necessary to connect the oscillator into the test circuit without the oscillator becoming an effective part of the circuit. One way to do

this is to use a high quality high ratio stepdown transformer between the oscillator and the circuit. This usually adds an extra resonant condition which can be designed to fall beyond the range of the test circuit resonant points and is therefore no problem. Another method is to use a resistor as shown in Fig. 10·16. This effectively isolates the oscillator from the test circuit and still produces a voltage variation in the circuit. The variable resistor is adjusted to prevent overloading and dis-

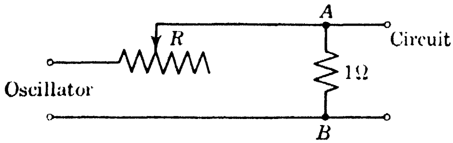


FIG. 10·16.

tortion. An oscillograpn connected across AB may be used to check the setting to see that a constant undistorted voltage is applied throughout the frequency range used. For inaccurate work, the coupling device may be eliminated.

After the circuit is hooked up with the oscillator, it is ready to test. Measurements may be made across any two points in the circuit to find resonant conditions as well as amplitudes, velocities, and accelera-

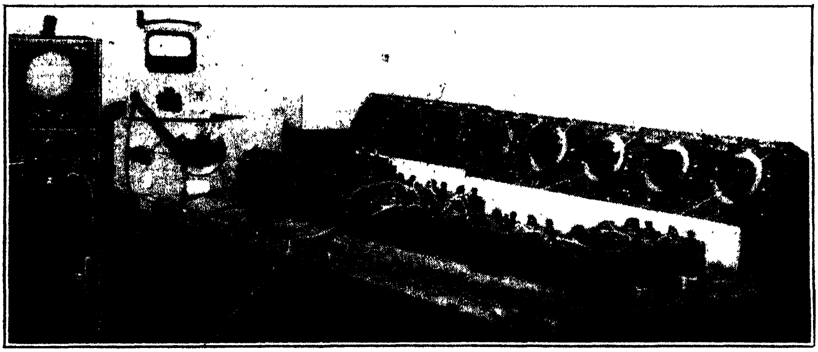


FIG. 10·17.

tions. Zero damping conditions are impossible because of the resistance inherent in any circuit. The measurements may be made most easily using a good electronic voltmeter having a frequency response as wide as the frequencies produced by the oscillator. This value may then be compared with the voltage across the coupling resistor to get a ratio of forces if desired. An alternative method of measuring is to use an oscillograph. This gives a visual picture whose amplitude can be measured.

It should be pointed out that it is often necessary to measure at several points to detect all the frequencies. This is because of insensitivity of instruments, of the nodal conditions where measurements are being made, or of the character of the circuit which produces motion in one portion of the system and very little in another. Figure 10·17 shows a typical setup using items mentioned in this section. Expensive calculating boards and analog installations are now also available for those who have frequent need for this type of equipment.

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ANSWERS

2·1. $f = 201$ cycles/min; $\tau = 0.00497$ min

2·5. $f = 248$ cycles/min; $v = 45.3$ in./sec

2·6. $G = 10{,}500{,}000$ lb/in.^{2}; $G = 12{,}000{,}000$ lb/in.^{2}

2·8. $k = \dfrac{k_1(k_2 + k_3)}{k_1 + k_2 + k_3}$

2·10. $k = \dfrac{G\pi}{32} \dfrac{d_1^4 d_2^4}{(d_1^4 L_2 + d_2^4 L_1)}$

2·15. $f = \dfrac{1}{2\pi}\sqrt{\dfrac{2kga^2}{WL^2} - \dfrac{g}{L}}$

2·16. $R = 39.1$ in.

2·17. $f = 2.21$ cycles/sec for water

2·18. $f = 6500$ cycles/min

2·19. $f = 31.5$ cycles/min

2·22. $f = \dfrac{1}{2\pi}\sqrt{\dfrac{kr^2}{I + \dfrac{W}{g}R^2}}$

2·23. $k_1 = 811$ lb/in.; $k_2 = 203$ lb/in.; $f_1 = 1338$ cycles/min; $f_2 = 670$ cycles/min

2·24. $f = \dfrac{1}{2\pi}\sqrt{\dfrac{Wr^2}{(R - r)\left(I + \dfrac{Wr^2}{g}\right)}}$

2·25. $f = \dfrac{1}{2\pi}\sqrt{\dfrac{ug}{a}}$

2·27. $f = \dfrac{60}{2\pi}\sqrt{\dfrac{3EIkg}{W(2kL^3 + 3EI)}}$

2·28. $f = \dfrac{60}{2\pi}\sqrt{\dfrac{gkb^2 12}{W(a^2 + 4b^2)}}$

2·29. $I_{cg} = 0.94$ lb-in.-sec^2

2·30. $I_{cg} = 56.9$ lb-in.-sec^2

2·31. $f = \dfrac{60}{2\pi}\sqrt{\dfrac{8kg}{3W_1 + 2W_2}}$

2·32. $W_s = 0.06W$; $W_s = 0.31W$

2·33. $f = 2820$ cycles/min

2·34. $f = 2230$ cycles/min

2·35. $f = \dfrac{1}{2\pi}\sqrt{\dfrac{W(R - a)}{I + \dfrac{W}{g}a^2}}$ cycles/sec

3·1. (*a*) $r = 1.11$ lb-sec/in.; (*b*) $f_{nd} = 1.75$ cycles/sec; (*c*) $f_n = 1.81$ cycles/sec; (*d*) $\delta = 1.64$; (*e*) $r_c = 4.4$ lb-sec/in.; (*f*) $x_2 = 0.388$ in.; (*g*) $F = 158.4$ lb

3·3. $f = 1.81$ cycles/sec; $\dfrac{x_2}{x_1} = 0.808$; $x_{10} = 0.236$ in.

3·4. $\dfrac{r}{r_c} = 0.05$; $r = 0.833$ lb-sec/in.

3·5. $f_n = 3.42$ cycles/sec; $x_2 = 0.395$ in.; $x_4 = 0$ in.
3·7. $f_{nd} = 1.18$ cycles/sec; $\theta_2 = 3.2°$
3·8. $x = 0.0269$ in.; $x = 0.088$ in.; $x = 0.00278$ in.
3·11. $x = -0.00107$ in.
3·12. $x = 3.26$ in.

3·13. $\dfrac{WL^2}{g}\dfrac{d^2\theta}{dt^2} + ra^2\dfrac{d\theta}{dt} + ka^2\theta = 0$

3·15. (*a*) $f_{nd} = 0.531$ cycle/sec; (*b*) $f_n = 0.532$ cycle/sec; (*c*) $\delta = 0.376$; (*d*) $r_c = 33.4$; (*e*) $\theta_2 = 6.87°$
3·16. $\theta = 0.0201$ radian; $s_s = 1110$ lb/in.^{2}
3·18. $x = \frac{3}{16}$ in.
3·19. $A = 50{,}400$ in./sec^2; $x = 1.42$ in.

3·20. $\dfrac{x_m}{\delta_{st}} = 0.224$

3·23. $f = 1290$ cycles/min; $x = 0.0033$ in.
3·24. $x_0 = 0.0008$ in.; $F = 0.470$ lb

4·1. $f = 98$ or 233 cycles/min
4·2. $f = 373$ or 170 cycles/min
4·3. $f = 2790$ and 1690 cycles/min
4·4. $f = 1360$ cycles/min
4·5. $f = 2250$ cycles/min
4·6. $f = 7420$ and 2970 cycles/min
4·14. $f = 12{,}700$ cycles/min; $f = 35{,}500$ cycles/min
4·15. $f = 165$ cycles/min
4·16. $f = 205$ cycles/min
4·17. Compare with 4.18.
4·18. $f = 277$ cycles/min; $f = 878$ cycles/min; $f = 1230$ cycles/min
4·19. $f = 325$ cycles/min
4·20. $f = 92$ cycles/min
4·21. $f = 265$ or 390 or 420 cycles/min

4·22. $f = \dfrac{1}{2\pi}\sqrt{\dfrac{4kg}{W}}$; $f = \dfrac{1}{2\pi}\sqrt{\dfrac{kg}{W}}$; $x = x_0 \cos\sqrt{\dfrac{kg}{W}}\,t$; $\theta = \theta_0 \cos\sqrt{\dfrac{4kg}{W}}\,t$

5·1. $k = 17.45$ lb/in.; T.R. = 0.0102
5·2. T.R. = 22.2%; $k = 10{,}000$ lb/in.
5·3. $k = 2360$ lb/in.
5·6. $\delta_{st} = 2.44$ in.; Isolation = 87.5%
5·7. T.R. = 10.5%; T.R. = 11.1%
5·8. $k = 189$ lb/in. for 100 rpm; T.R. = 19.05%; T.R. = 12.5%; T.R. = 4.17%
5·9. SAE F-6 84 sq in. total
5·10. T.R. = 226%; T.R. = 12.8%

6·1. $f = 25.4$ cycles/sec
6·2. $f = 1450$ or 3020 cycles/min
6·4. $f = 1660$ or 2810 cycles/min
6·5. $f = 550$ cycles/min; $f = 22{,}300$ cycles/min
6·6. $f = 22{,}300$ cycles/min
6·7. $f = 7390$ cycles/min
6·8. $f = 2390$ cycles/min; $f = 10{,}500$ cycles/min
6·9. (See 6·8)

7·1. $f = 698$ cycles/min
7·2. $f = 909$ cycles/min
7·3. $f = 2600$ cycles/min
7·4. $f = 2650$ cycles/min
7·5. $f = 960$ cycles/min
7·9. $W_b = 0.465W$
7·12. $f = 5.73$ cycles/sec
7·16. $f = 980$ cycles/min

8·2. db $= 74.19$
8·4. db $= 76.6$
8·5. db $= 5.71$
8·7. db $= 61.6$
8·10. db $= 28.8$
8·11. db $= 32.4$
8·13. $s = 890$ sabins
8·14. Area $= 5250$ ft

9·1. $v = 3.54\overline{\backslash 27° - 35'}$ in./sec; $x = 0.094\overline{\backslash 117° - 35'}$ in.
9·2. $I_1 I_2 \omega^4 - \omega^2[k_{t2}(I_1 + I_2) + k_{t1}I_2] + k_{t2}k_{t1} = 0$
9·4.
$$\omega\left[\omega^6 \frac{I_1 I_2 I_3 I_4}{k_{t1}k_{t2}k_{t3}} - \omega^4\left(\frac{(I_1 + I_2)I_3 I_4}{k_{t2}k_{t3}} + \frac{(I_2 + I_3)I_1 I_4}{k_{t1}k_{t3}} + \frac{(I_3 + I_4)I_1 I_2}{k_{t1}k_{t2}}\right)\right.$$
$$+ \omega^2\left(\frac{(I_1 + I_2 + I_3)I_4}{k_{t3}} + \frac{(I_1 + I_2)(I_3 + I_4)}{k_{t2}} + \frac{(I_2 + I_3 + I_4)I_1}{k_{t1}}\right)$$
$$\left. - (I_1 + I_2 + I_3 + I_4)\right] = 0$$

9·5. $f = \dfrac{1}{2\pi}\sqrt{\dfrac{k_{t1}(I_1 + I_2)}{I_1 + I_2}}$
9·6. $\theta_f = -0.0683$ radian; $\theta_c = -0.687$ radian; $\theta_a = -0.0690$ radian; $s_s = 880$ psi and 670 psi
9·7. $\theta = -0.0685$ radian; $\theta = -0.0690$ radian; $\theta = -0.0688$ radian; $s_s = 404$ lb/in.^{2}; $s_s = 665$ lb/in.^{2}
9·8. $\theta_f = -0.0000315$ radian; $\theta_a = 0.00028$ radian; $\theta_c = -0.000122$ radian; $s_s = 610$ lb/in.^{2}

9·9. $z = 0.668 - 0.937i$; $x = 0.0366\overline{\backslash 144.5°}$ in.
9·10. $\dfrac{W_1}{g}\dfrac{W_2}{g}\omega^4 - \omega^2\left(\dfrac{W_1}{g}(k_2 + k_3) + \dfrac{W_2}{g}(k_1 + k_2)\right) + k_1k_2 + k_2k_3 + k_1k_3 = 0$;
$x_1 = 0.0775$ in.; $x_2 = 0.0268$ in.

INDEX